(m)-Associahedra and (-m)-Noncrossing Partition Polynomials as Moments of Umbral Inverse Appell Polynomial Sequences

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Reciprocity of the (m)-associahedra and (-m)-noncrossing partitions polynomials

In several recent posts at my WordPress mini-ArXiv "Shadows of Simplicity", I've explored the algebra and combinatorics associated with the sets $[A^{(m)}]$ of (m)-associahedra partition polynomials (ParPs) $A_n^{(m)}(u_1, ..., u_n)$ and the sets $[N^{(m)}]$ of (m)-noncrossing partitions ParPs $N_n^{(m)}(u_1, ..., u_n)$, or (m)-Narayana ParPs, where m is any integer. Explicit formulas for the coefficients are available in other posts and will be available in a forthcoming update of a pdf. The objective of this pdf is to present some general identities related to the calculus of Appell Shefer polynomial sequences, which is briefly presented in the appendix, and to illustrate and corroborate the results.

The sets $[N^{(m)}] = [N]^{(m)}$ and the involutive $[A^{(m)}]$ are related via the substitution identity (right set into left)

$$[A^{(m)}] = [N^{(m)}][A^{(0)}] = [N^{(m)}][R] = [N]^{(m)}[R],$$

and

$$[A^{(m)}][A^{(0)}] = [A^{(m)}][R] = [N^{(m)}] = [N]^{(m)},$$

where $[A^{(0)}] = [R]$ are the reciprocal ParPs, which are the set of ParPs $R_n(u_1, ..., u_n)$ giving the coefficients of the multiplicative inverse, or reciprocal, of the o.g.f. $O(x) = 1 + u_1x + u_2x^2 + \cdots$.

Taking the substitutional inverse of both sides of the above identities gives

$$[A^{(m)}] = [A^{(0)}][N^{(-m)}] = [R][N^{(-m)}] = [R][N]^{-m},$$

and

$$[A^{(m)}][A^{(0)}] = [R][A^{(m)}] = [N^{(-m)}] = [N]^{(-m)}.$$

This implies

$$A^{(m)}(t)N^{(-m)}(t) = 1,$$

where

$$A^{(m)}(t) = 1 + A_1^{(m)}(u_1)t + A_2^{(m)}(u_1, u_2)\frac{t^2}{2!} + \cdots$$

and

$$N^{(-m)}(t) = 1 + N_1^{(-m)}(u_1)t + N_2^{(-m)}(u_1, u_2)t^2 + \cdots .$$

Then the series $A^{(m)}(t)$ and $N^{(-m)}(t)$ stand in relation to each other, aside from sign conventions, as do the complete homogeneous symmetric function series $H(t)$ and the elementary symmetric function series $E(t)$ of symmetric function theory.

The calculus of Appell Sheffer sequences may also be applied, being careful about normalization factors. For a short review of the basics of Appell Sheffer sequences see the appendix. For more info of Appell sequences, scan my posts at my WordPress site "Shadows of Simplicity", my MathOverflow posts, and OEIS A133314.

As demonstrated for the log Narayana ParPs in the previous post, with the logarithmic derivatives

$$LN^{(-m)}(t) = D_t \ln[N^{(-m)}(t)]$$

and

$$LA^{(m)}(t) = D_t \ln[A^{(m)}(t)],$$

the raising ops for generating the ParPs $n!N_n^{(-m)}$ and $n!A_n^{(m)}$ are

$$R_{N^{(-m)}} = x + D_{t=D_x} \ln[N^{(-m)}(t)] = x + LN^{(-m)}(D_x) = x - LA^{(m)}(D_x)$$

and

$$R_{A^{(m)}} = x + D_{t=D_x} \ln[A^{(m)}(t)] = x + LA^{(m)}(D_x) = x - LN^{(-m)}(D_x).$$

Note that $A^{(-|m|)}(t)N^{(|m|)}(t) = 1$ is consistent with all the $N_n^{|m|}$ for $n > 0$ having all positive integer coefficients and all the $A_n^{-|m|}$ having all negative integer coefficients. In addition, $\ln[N^{(|m|)}] = -\ln[A^{(-|m|)}]$, implies the coefficients of the logarithmic derivative $LN^{|m|}(t)$ are all positive integers and. therefore, those of $LA^{-|m|}(t)$ are all negative.

The series being reciprocals also implies for all integer m that

$$\sum_{k=0}^{n} A_k^{(m)} N_{n-k}^{(-m)} = \delta_n,$$

which can be translated into a mutual recursion relation.

More generally, with the renormalizations $\bar{N}_n^{(m)} = n! N_n^{(m)}$ and $\bar{A}_n^{(m)} = n! A_n^{(m)}$, the full panoply of relations I've presented for umbral inverse pairs of Appell polynomials sequences and their moments using the moments $\bar{N}_n^{(-m)}$ and $\bar{A}_n^{(m)}$--mutual moment recursion relations, orthogonal matrix relations, extended group relations, and more. Alternatively, symmetric function theory involving the Faber polynomials and elementary Schur polynomials can be applied.

Illustration / spot checks for the raising ops

For $[A^{(2)}]$, the first few ParPs are

$A_0^{(2)} = 1$,

$A_1^{(2)} = -u_1$,

$A_2^{(2)} = -u_2 + 3u_1^2$,

$A_3^{(2)} = -u_3 + 8u_1u_2 - 12u_1^3$,

$A_4^{(2)} = -u_4 + 10u_1u_3 + 5u_2^2 - 55u_1^2u_2 + 55u_1^4$, so

$A^{(2)}(t) = 1 - u_1 t + (-u_2 + 3u_1^2)t^2 + (-u_3 + 8u_1 u_2 - 12u_1^3)t^3$

$+(-u_4 + 10u_1 u_3 + 5u_2^2 - 55u_1^2 u_2 + 55u_1^4)t^4 + \cdots$

and

$LA^{(2)}(D_x) = D_{t=D_x} \ln(A^{(2)}(t))$

$= -u_1 + (5u_1^2 - 2u_2)D_x + (-28u_1^3 + 21u_2 u_1 - u_3)D_x^2$

$+(165u_1^4 - 180u_2 u_1^2 + 36u_3 u_1 + 18u_2^2 - 4u_4)D_x^3 + \cdots$.
.

For $[N^{(-2)}]$, the first few ParPs are

$N_0^{(-2)} = 1$,

$N_1^{(-2)} = u_1$,

$N_2^{(-2)} = -2u_1^2 + u_2$,

$N_3^{(-2)} = 7u_1^3 - 6u_2 u_1 + u_3$,

$N_4^{(-2)} = -30u_1^4 + 36u_2 u_1^2 - 8u_3 u_1 - 4u_2^2 + u_4$,

so

$N^{(-2)}(t) = 1 + u_1 t + (-2u_1^2 + u_2)t^2 + (7u_1^3 - 6u_2 u_1 + u_3)t^3$

$+(-30u_1^4 + 36u_2 u_1^2 - 8u_3 u_1 - 4u_2^2 + u_4)t^4 + \cdots$

and

$LN^{(-2)}(D_x) = D_{t=D_x} \ln(N^{(-2)(t)})$

$= u_1 + (2u_2 - 5u_1^2)D_x + (28u_1^3 - 21u_2u_1 + u_3)D_x^2$

$+(-165u_1^4 + 180u_2u_1^2 - 36u_3u_1 - 18u_2^2) + 4u_4)D_x^3 + \cdots$,

consistent with theory up to the given order with $LN^{(-2)}(D_x) = -LA^{(2)}(D_x)$.

For the pair $[A^{(-2)}]$ and $[N]^2$:

$A_0^{(-2)} = 1$,

$A_1^{(-2)} = -u_1$,

$A_2^{(-2)} = -(u_2 + u_1^2)$,

$A_3^{(-2)} = -(2u_1^3 + 4u_2u_1 + u_3)$,

$A_4^{(-2)} = -(5u_1^4 + 15u_2u_1^2 + 6u_3u_1 + 3u_2^2 + u_4)$,

so

$A^{(-2)}(t) = 1 - u_1t - (u_2 + u_1^2)t^2 - (2u_1^3 + 4u_2u_1 + u_3)t^3$

$-(5u_1^4 + 15u_2u_1^2 + 6u_3u_1 + 3u_2^2 + u_4)t^4 + \cdots$

and

$LA^{(2)}(D_x) = D_{t=D_x} \ln(A^{(-2)}(t))$

$= -u_1 - (6u_1^2) + 2u_2)D_x - (10u_1^3 + 15u_2u_1 + 3u_3)D_x^2$

$-(35u_1^4 + 84u_2u_1^2 + 28u_3u_1 + 14u_2^2 + 4u_4)D_x^3 + \cdots$.

.

$N_0^{(2)} = 1$,

$N_1^{(2)} = u_1$,

$N_2^{(2)} = 2u_1^2 + u_2$,

$N_3^{(2)} = 5u_1^3 + 6u_2u_1 + u_3$,

$N_4^{(2)} = 14u_1^4 + 28u_2u_1^2 + 8u_3u_1 + 4u_2^2 + u_4$,

so

$N^{(2)}(t) = 1 + u_1t + (2u_1^2 + u_2)t^2 + (5u_1^3 + 6u_2u_1 + u_3)t^3$

$+(14u_1^4 + 28u_2u_1^2 + 8u_3u_1 + 4u_2^2 + u_4)t^4 + \cdots$

and

$Ln^{(2)}(D_x) = u_1 + (3u_1^2 + 2u_2)D_x + (10u_1^3 + 15u_2u_1 + 3u_3)x^2$

$+(35u_1^4) + 84u_2u_1^2 + 28u_3u_1 + 14u_2^2) + 4u_4)D_x^3 + \cdots$.

The computations are in agreement to this order with the theoretical conclusion $-LA^{(-2)}(D_x) = LN^{(2)}(D_x)$.

Let's work out one short sequence raisings for $[A^{(2)}]$:

$R_A^{(2)} = x + LA^{(2)}(D_x)$

$= x - u_1 + (5u_1^2 - 2u_2)D_x + (-28u_1^3 + 21u_2u_1 - 3u_3)D_x^2$

$+(165u_1^4 - 180u_2u_1^2 + 36u_3u_1 + 18u_2^2 - 4u_4)D_x^3 + \cdots$,

so

$R_A^{(2)}\ 1|_{x=0} = (x - u_1)|_{x=0} = -u_1 = 1!A_1^{(2)}(u_1)$,

$R_A^{(2)}(x - u_1)|_{x=0} = (x - u_1)^2 + (5u_1^2 - 2u_2)|_{x=0} = 6u_1^2 - 2u_2$

$= 2!(3u_1^2 - u_2) = 2!A_2^{(2)}(u_1, u_2)$,

and

$$R_A^{(2)}[(x-u_1)^2+(5u_1^2-2u_2)]|_{x=0}$$

$$=(x-u_1)[(x-u_1)^2+(5u_1^2-2u_2)]+(5u_1^2-2u_2)[2((x-u_1))]$$

$$+(-28u_1^3+21u_2u_1-3u_3)2|_{x=0}$$

$$=(-u_1)[(-u_1)^2+(5u_1^2-2u_2)]+(5u_1^2-2u_2)[2((-u_1))]$$

$$+(-28u_1^3+21u_2u_1-3u_3)2$$

$$=-72u_1^3+48u_2u_1-6u_3=3!(-12u_1^3+8u_2u_1-u_3)$$

$$=3!A_1^{(2)}(u_1,u_2,u_3).$$

Reciprocities of reductions

Reprising, the most refined reciprocal identity for the associahedra and noncrossing ParPs is

$$A^{(m)}(t)=\frac{1}{N^{(-m)}(t)}.$$

With all $c_n=1$, this reduces to the reciprocal relation between the sums of the coefficients of the ParPs

$$SA^{(m)}(t)=\frac{1}{SN^{(-m)}(t)}.$$

Example:

The row sums, $[SA^{(3)}]$, of $[A^{(3)}]$ are $(1,-1,3,-12,55,...)$.

The row sums, $[SN^{(-3)}]$, of $[N^{(-3)}]$ are $(1,1,-2,7,-30,143,...)$, so

$$\sum_{k=0}^{n} SA_{n-k}^{(3)} SN_k^{(k)} = \delta_n.$$

Spot check:

$(1, -1, 3) * (-2, 1, 1) = -2 - 1 + 3 = 0$

$(1, -1, 3, -12) * (7, -2, 1, 1) = 7 + 2 + 3 - 12 = 0.$

Extension of the identities:

Reprising, the central generalized face-h-polynomial identity (F-H identity) for the sets of ParPs is

$[A^{(m)}] = [N^{(m)}][R] = [N^{(m)}][A^{(0)}]$

and, symmetrically,

$[A^{(m)}][R] = [A^{(m)}][A^{(0)}] = [N^{(m)}].$

Since $[A^{(m)}|^2 = [A^{(m)}]([A^{(m)}]^{-1} = [I]$, the substitutional identity, and $[N^{(m)}] = [N]^{-m}$, we have at our disposal the reciprocal identity (R-identity)

$[A^{(m)}] = [R][N]^{-m} = [R][N^{(-m)}].$

Now is a good point to divest the terminology in previous posts and other literature of peoples' names and attach it to geometric constructs. I already let A denote associahedra and N noncrossing partitions, but, in my previous posts, influenced by other authors, I've used $FFC_n^{(m)}(m)$ to denote free (m)-Fuss-Catalan numbers and $FFN_n^{(m)}$ to denote free (m)-Fuss-Narayana numbers, where free means possibly negative as well as positive integers. Instead, in the following, I use $DA_n^{(m)}$ and $DN_n^{(m)}$ as more informative, mnemonic descriptors of these sequences of numbers with $DN_n^{(m)}$ being the coefficient of the 'diagonal' component of $N_n^{(m)}(u_1, ..., u_n)$, i.e., the coefficient of the u_1^n monomial, the highest possible order monomial, which can be negative as well as positive, and $D_n^{(m)} = DA_n^{(m)}$ being the coefficient of the ‘diagonal’ of $A_n^{(m)}(u_1, ..., u_n)$. The term diagonal is derived from the arrangement in a triangular matrix of the coefficients of sequences of mono-variable Sheffer polynomials $Shef_n(x)$ of order n along row n in ascending order with the coefficient of x^n on the diagonal.

The F-H identity implies

$$DA^{(m)}(t) = SN^{(m)}(-t)$$

and

$$SA^{(m)}(t) = DN^{(m)}(-t)$$

and the R-identity implies

$$A^{(m)}(t) = \frac{1}{N^{(-m)}(t)}.$$

Then,

$$SA^{(m)}(t) = \frac{1}{SN^{(-m)}(t)}$$

$$= DN^{(m)}(-t) = \frac{1}{DA^{(-m)}(-t)}.$$

Analysis of the reductions implies also

$$DA^{(m)}(t) = DN^{(m+1)}(-t).$$

Consequently, we have the four equivalent identities for any integer m

$$SA^{(m)}(t) = \frac{1}{SN^{(-m)}(t)}$$

$$= DN^{(m)}(-t) = \frac{1}{DA^{(-m)}(-t)}$$

$$= SN^{(m-1)}(-t) = \frac{1}{SA^{(-(m-1))}(-t)}$$

$$= DA^{(m-1)}(t) = \frac{1}{DN^{(-(m-1))}(t)}.$$

Illustrations / spot checks:

(The numbers below are obtained directly from the ParPs rather than any formulas for their reductions, but formulas for the reductions were used to verify the full sequences are indeed the OEIS entries.)

OEIS $A001764 = (1, 1, 3, 12, 55, ...)$ and, using the equality slightly loosely as equivalence,

$[SA^{(3)}] = (1, -1, 3, -12, 55, ...) = SA^{(3)}(t)$,

$[DN^{(3)}] = (1, 1, 3, 12, 55, 273, ...) = DN^{(3)}(t) = SA^{(3)}(-t)$,

$[SN^{(2)}] = (1, 1, 3, 12, 55, 273, ...) = SN^{(2)}(t) = SA^{(3)}(-t)$,

$[DA^{(2)}] = (1, -1, 3, -12, 55, ...) = DA^{(2)}(t) = SA^{(3)}(t)$.

OEIS $A006013 = (1, 2, 7, 30, 143, ...)$ and

$[SN^{(-3)}] = (1, 1, -2, 7, -30, 143, ...) = SN^{(-3)}(t)$,

$[DA^{(-3)}] = (1, -1, -2, -7, -30, -143, ...) = DA^{(-3)}(t) = SN^{(-3)}(-t)$,

$[SA^{(-2)}] = (1, -1, -2, -7, -30, -143, ...) = SA^{(-2)}(t) = SN^{(-3)}(-t)$.

$[DN^{(-2)} = (1, 1, -2, 7, -30, 143, ...) = DN^{(-2)}(t) = SN^{(-3)}(t)$.

Then

$$SA^{(3)}(t) = \frac{1}{SN^{(-3)}(t)}$$

$$= DN^{(3)}(-t) = \frac{1}{DA^{(-3)}(-t)}$$

$$= SN^{(2)}(-t) = \frac{1}{SA^{(-2)}(-t)}$$

$= DA^{(2)}(t) = \frac{1}{DN^{(-2)}(t)}$.

Next lower order:

OEIS $A000108 = (1, 1, 2, 5, 14, 42, 132, ...)$,

$[SA^{(2)}] = SA^{(2)}(t) = (1, -1, 2, -5, 14, ...)$,

$[DN^{(2)}] = (1, 1, 2, 5, 14, 42, ...) = DN^{(2)}(t) = SA^{(2)}(-t)$,

$[SN] = (1, 1, 2, 5, 14, 42, ...) = SN^{(1)}(t) = SA^{(2)}(-t)$,

$[DA] = (1, -1, 2, -5, 14, ...) = DA^{(1)}(t) = SA^{(2)}(t)$,

$(1, A000108) = (1, 1, 1, 2, 5, 14, 42, 132, ...)$,

$SN^{(-2)}(t) = (1, 1, -1, 2, -5, 14, ...)$,

$[DA^{(-2)}] = (1, -1, -1, -2, -5, -14, ...) = DA^{(-2)}(t) = SN^{(-2)}(-t)$,

$[SA^{(-1)}] = (1, -1, -1, -2, -5, -14, ...) = SA^{(-1)}(t) = SN^{(-2)}(-t)$,

$[DN^{(-1)}] = (1, 1, -1, 2, -5, 14, ...) = DN^{(-1)}(t) = SN^{(-2)}(t)$.

So,

$SA^{(2)}(t) = \frac{1}{SN^{(-2)}(t)}$

$= DN^{(2)}(-t) = \frac{1}{DA^{(-2)}(-t)}$

$= SN^{(1)}(-t) = \frac{1}{SA^{(-1)}(-t)}$

$= DA^{(1)}(t) = \frac{1}{DN^{(1)}(t)}$.

Next order check:

$$(1,1,1,1,1,...) = \frac{1}{1-t},$$

$$SA^{(1)}(t) = (1,-1,1,-1,1,...),$$

$$DN^{(1)}(t) = (1,1,1,1,...) = SA^{(1)}(-t),$$

$$SN^{(0)}(t) = (1,1,1,1,...) = SA^{(1)}(-t),$$

$$DA^{(0)}(t) = (1,-1,1,-1,1,...) = SA^{(1)}(t).$$

$$(1,-1,0,0,0,0,...) = 1-t,$$

$$SN^{(-1)}(t) = (1,1,0,0,0,...),$$

$$DA^{(-1)}(t) = (1,-1,0,0,0,...) = SN^{(-1)}(-t),$$

$$SA^{(0)}(t) = (1,-1,0,0,0,,...) = SN^{(-1)}(-t),$$

$$DN^{(0)}(t) = (1,1,0,0,0,...) = SN^{(-1)}(t).$$

Next order is essentially the reciprocal of the one just above as are the others below:

$$(1,-1,0,0,0,0,...) = 1-t.$$

$$SA^{(0)}(t) = (1,-1,0,0,0,...),$$

$$DN^{(0)}(t) = (1,1,0,0,0,...) = SA^{(0)}(-t),$$

$$SN^{(-1)}(t) = (1,1,0,0,0,...) = SA^{(0)}(-t),$$

$$DA^{(-1)}(t) = (1,-1,0,0,0,...) = SA^{(0)}(t).$$

$(1,1,1,1,1,...) = \frac{1}{1-t}$.

$SN^{(0)}(t) = (1,1,1,1,...)$.

$DA^{(0)}(t) = (1,-1,1,-1,1,...) = SN^{(0)}(-t)$,

$SA^{(1)}(t) = (1,-1,1,-1,1,...) = SN^{(0)}(-t)$,

$DN^{(1)}(t) = (1,1,1,1,...) = SN^{(0)}(t)$..

$SA^{(0)}(t) = \frac{1}{SN^{(0)}(t)}$

$= DN^{(0)}(-t) = \frac{1}{DA^{(0)}(-t)}$

$= SN^{(-1)}(-t) = \frac{1}{SA^{(1)}(-t)}$

$= DA^{(-1)}(t) = \frac{1}{DN^{(1)}(t)}$.

Next order:

$(1, A000108) = 1,1,1,2,5,14,42,...)$.

$SA^{(-1)}(t) = (1,-1,-1,-2,-5,-14)$,

$DN^{(-1)}(t) = (1,1,-1,2,-5,14,...) = SA^{(-1)}(-t)$,

$SN^{(-2)}(t) = (1,1,-1,2,-5,14,...) = SA^{(-1)}(-t)$,

$DA^{(-2)}(t) = (1,-1,-1,-2,-5,-14,...) = SA^{(-1)}(t)$.

$A000108 = (1,1,2,5,14,42,...)$.

$SN^{(1)}(t) = (1, 1, 2, 5, 14, ...)$.

$DA^{(1)}(t) = (1, -1, 2, -5, 14, ...) = SN^{(1)}(-t)$

$SA^{(2)}(t) = (1, -1, 2, -5, 14, ...) = SN^{(1)}(-t)$

$DN^{(2)}(t) = (1, 1, 2, 5, 14, ...) = SN^{(1)}(t)$.

$$SA^{(-1)}(t) = \frac{1}{SN^{(1)}(t)}$$

$$= DN^{(-1)}(-t) = \frac{1}{DA^{(1)}(-t)}$$

$$= SN^{(-2)}(-t) = \frac{1}{SA^{(2)}(-t)}$$

$$= DA^{(-2)}(t) = \frac{1}{DN^{(2)}(t)}.$$

Next order:

OEIS $A006013 = (1, 2, 7, 30, 143, ...)$.

$SA^{(-2)}(t) = (1, -1, -2, -7, -30, -143, ...)$,

$DN^{(-2)}(t) = (1, 1, -2, 7, -30, 143, ...) = SA^{(-2)}(-t)$,

$SN^{(-3)}(t) = (1, 1, -2, 7, -30, 143, ...) =$,

$DA^{(-3)}(t) = (1, -1, -2, -7, -30, -143, ...) =$.

OEIS $A001764 = (1, 1, 3, 12, 55, 273, ...)$,

$SN^{(2)}(t) = (1, 1, 3, 12, 55, 273, ...) =$,

$DA^{(2)}(t) = (1, -1, 3, -12, 55, ...) = SN^{(2)}(-t)$,

$SA^{(3)}(t) = (1, -1, 3, -12, 55, ...)$,

$DN^{(3)}(t) = (1, 1, 3, 12, 55, 273, ...)$.

$$SA^{(-2)}(t) = \frac{1}{SN^{(2)}(t)}$$

$$= DN^{(-2)}(-t) = \frac{1}{DA^{(2)}(-t)}$$

$$= SN^{(-3)}(-t) = \frac{1}{SA^{(3)}(-t)}$$

$$= DA^{(-3)}(t) = \frac{1}{DN^{(3)}(t)}.$$

This is the reciprocal of the top example.

Further identities from sextuplets (for $m \geq 1$):

From my doc "Reciprocal Sheffer Sextuplets and Conjugation Isomorphism" with the compositional inverse pair

$$h(t) = t\ A^{(m)}(t) = t\ (1 + A_1^{(m)}(c_1)t + A_2^{(m)}(c_1, c_2)t^2 + A_3^{(m)}(c_1, c_2, c_3)t^3 + \cdots)$$

and

$$h^{(-1)}(t) = t\ O^{(m)}(t) = t\ (1 + c_1 t^m + c_2 t^{2m} + c_3 t^{3m} + \cdots),$$

the umbral inverse **binomial Sheffer pair** of the sextuplet is defined by

1)

$$e^{tA^{(m)}(t)x} = e^{u.(x)t}$$

and

$e^{tO^{(m)}(t)x} = e^{v.(x)t}$, elementary Schur functions (A130561).

The two **umbral inverse Appell Sheffer pairs** are defined by

2)

$\frac{1}{A^{(m)}(t)} e^{xt} = N^{(-m)}(t) e^{xt} = e^{p.(x)t}$, a (-m)-Narayana Appell sequence

and

$O^{(m)}(t) e^{xt} = e^{q.(x)t}$

3)

$A^{(m)}(t) e^{xt} = e^{r.(x)t}$

and

$\frac{1}{O^{(m)}(t)} e^{xt} = R^{(m)}(t) e^{xt} = e^{s.(x)t}$, reciprocal Appell sequence.

Then generally

$p_n(x) = u_n(q.(v.(x)))$

and

$r_n(x) = u_n(s.(v.(x)))$.

Specialized to $m = 1$, the first few ParPs of $[A]$ are

$A_0 = 1$,

$A_1 = -c_1$,

$$A_2 = 2c_1^2 - c_2,$$

$$A_3 = -5c_1^3 + 5c_1c_2 - c_3,$$

$$A_4 = 14c_1^4 - 21c_1^2c_2 + 6c_1c_3 + 3c_2^2 - c_4,$$

and, for the formalism of the sextuplet, this specialization gives

$$u(t) = t\ A(t) = t + A_1t^2 + A_2t^3 + A_3t^3 + \cdots$$

$$= t - c_1t^2 + (2c_1^2 - c_2)t^3 + (-5c_1^3 + 5c_1c_2 - c_3)t^4$$

$$+(14c_1^4 - 21c_1^2c_2 + 6c_1c_3 + 3c_2^2 - c_4)t^5 + \cdots.$$

Then the Taylor series expansion of

$$e^{u.(t)}x = \exp(A(t)x)$$

generates the first few ParP coefficients

$$u_0 = 1,$$

$$u_1 = x,$$

$$u_2 = x^2 - 2c_1x,$$

$$u_3 = x^3 - 6x^2c_1 + (12c_1^2 - 6c_2)x,$$

$$u_4 = x^4 - 12c_1x^3 + (60c_1^2 - 24c_2)x^2 + (-120c_1^3 + 120c_1c_2 - 24c_3)x,$$

$$u_5 = x^5 - 20x^4c_1 + (180c_1^2 - 60c_2)x^3 + (-840c_1^3 + 720c_1c_2 - 120c_3)x^2$$

$$+(1680c_1^4 + 360c_2^2 - 2520c_1^2c_2 + 720c_1c_3 - 120c_4)x.$$

.

With

$$t\ O(t) = t + c_1t^2 + c_2t^3 + c_3t^4 + \cdots,$$

$e^{tO(t)x} = \exp[v.(x)t]$ generates

$v_0 = 1$,

$v_1 = x$,

$v_2 = 2c_1x + x^2$,

$v_3 = 6c_2x + 6c_1x^2 + x^3$,

$v_4 = 24c_3x + (12c_1^2 + 24c_2)x^2 + 12c_1x^3 + x^4.$

These are a variant of A130561, essentially the elementary Schur polynomials.

The first few ParPs of $[N^{(-1)}]$ are

$N_0^{(-1)} = 1$,

$N_1^{(-1)} = c_1$,

$N_2^{(-1)} = c_2 - c_1^2$,

$N_3^{(-1)} = c_3 - 3c_2c_1 + 2c_1^3$,

$N_4^{(-1)} = c_4 - 2c_2^2 - 4c_3c_1 + 10c_2c_1^2 - 5c_1^4,$

and, from the formalism of Appell sequences sketched below,

$$p_n(x) = (p.(x))^n = ((.)!N.^{(-1)} + x)^n = \sum_{k=0}^{n} \binom{n}{k} k! N_k^{(-1)} x^{n-k}.$$

Similarly,

$$q_n(x) = ((.)!c. + x)^n = \sum_{k=0}^{n} \binom{n}{k} k! c_k x^{n-k},$$

$$r_n(x) = ((.)!A. + x)^n = \sum_{k=0}^{n} \binom{n}{k} k! A_k x^{n-k},$$

and

$$s_n(x) = ((.)!R. + x)^n = \sum_{k=0}^{n} \binom{n}{k} k! R_k x^{n-k}$$.

with the first few ParPs of $[R]$ being

$R_0 = 1$

$R_1 = -c_1,$

$R_2 = c_1^2 - c_2,$

$R_3 = -c_1^3 + 2c_1c_2 - c_3,$

$R_4 = c_1^4 - 3c_1^2c_2 + 2c_1c_3 + c_2^2 - c_4,$

which is independent of m.

Then the further specialization of the conjugation relation implies

$p_n(0) = n!N_n^{(-1)} = u_n(q.(0)) = u_n(d.)$, where $d_n = n!O_n = n!c_n$.

For example,

$p_2(0) = u_2(d.) = d_2 - 2c_1d_1$

$= 2!c_2 - 2c_1^2 = 2(c_2 - c_1^2 = 2!N_2^{(-1)}(c_1, c_2)$

and

$p_3(0) = u_3(d.) = d_3 - 6d_2c_1 + (12c_1^2 - 6c_2)d_1$

$= 3!c_3 - 6(2!c_2)c_1 + (12c_1^2 - 6c_2)c_1 = 12c_1^3 - 18c_2c_1 + 6c_3$

$= 3!(2c_1^3 - 3c_2c_1 + c_3) = 3!N_3^{(-1)}(c_1, c_2, c_3).$

Similarly,

$r_n(0) = n!A_n = u_n(s.(v.(0))) = u_n(d.))$ with $d_n = n!R_n = n!A_n^{(0)}$,

and

$= 3!R_3 - 6(2!R_2)c_1 + (12c_1^2 - 6c_2)(R_1)$

$= 3!(-c_1^3 + 2c_1c_2 - c_3) - 6(2!(c_1^2 - c_2))c_1 + (12c_1^2 - 6c_2)(-c_1)$

$= -30c_1^3 + 30c_2c_1 - 6c_3 = 3!(-5c_1^3 + 5c_2c_1 - 1c_3) = 3!A_3(c_1, c_2, c_3).$

The conjugations also imply

$v_n(p.(u.(x))) = q_n(x)$

and

$v_n(r.(u.(x))) = s_n(x),$

so

$v_n(p.(0)) = q_n(0) = n!c_n = v_n((.)!N.^{(-1)}) = v_n(d.)$ with $d_n = n!N_n^{(-1)}$

and

$v_n(r.(0)) = s_n(0) = n!R_n = v_n((.)!A.) = v_n(d.),$ with $d_n = n!A_n$

With $d_n = n!N_n^{(-1)}$,

$v_3(d.) = 6c_2d_1 + 6c_1d_2 + d_3$

$= 6c_2N_1^{(-1)} + 6c_1(2!N_2^{(-1)}) + 3!N_3^{(-1)}$

$= 6c_2c_1 + 6c_1(2!(c_2 - c_1^2)) + 3!(c_3 - 3c_2c_1 + 2c_1^3)$

$= 3!c_3$

With $d_n = n!A_n$,

$v_v(r.(0)) = s_3(0) = 3!R_3 = v_3((.)!A.) = v_3(d.).$

$v_3(d.) = 6c_2d_1 + 6c_1d_2 + d_3$

$= 6c_2A_1 + 6c_1(2!A_2) + 3!A_3$

$= 6c_2(-c_1) + 6c_1(2!(2c_1^2 - c_2)) + 3!(-5c_1^3 + 5c_1c_2 - c_3)$

$= -6c_1^3 + 12c_2c_1 - 6c_3 = 3!(-c_1^3 + 2c_2c_1 - c_3) = 3!R_3$.

An aside: Two other identities of the sextuplet are

$u'_{n+1}(x) = (n+1)r_n(u.(x))$

and

$v'_{n+1}(x) = (n+1)q_n(v.(x)),$

so, since $u_n(0) = \delta_n = v_n(0)$,

$u'_{n+1}(0) = (n+1)r_n(u.(0)) = (n+1)r_n(\delta.)$

and

$v'_{n+1}(0) = (n+1)q_n(v.(0)) = (n+1)q_n(\delta.),$

where $\delta.^k = \delta_k = 0^k$, is the Kronecker delta, which vanishes except for $\delta_0 = 1$.

For our specialization,

$u'_{n+1}(0) = (n+1)r_n(u.(0)) = (n+1)r_n(\delta.) = (n+1)r_n(0)$

becomes

$u'_{n+1}(0) = (n+1)!A_n$,

and from the examples of the series above for $u_n(x)$ and $A - n(x$ we see this formula corroborated, e.g.,

$u'_4(0) = (-120c_1^3 + 120c_1c_2 - 24c_3) = 3!A_4.$

The second related identity

$$v'_{n+1}(0) = (n+1)q_n(v.(0)) = (n+1)q_n(\delta.) = (n+1)q_n(0),$$

becomes

$v'_{n+1}(0) = (n+1)!c_n$, which is corroborated by the examples above for $v_n(x)$.

Appendix: Basics of Appell Sheffer polynomial sequences

Appell polynomials have the form

$$(\mathbb{A}.(x))^n = \mathbb{A}_n(x) = (\mathbb{A}. + x)^n = \sum_{k=0}^{n} \binom{n}{k} \mathbb{A}_{n-k} x^k$$

with moments $(\mathbb{A}.)^k = \mathbb{A}_k$ with $\mathbb{A}_0 = 1$.

The e.g.f., with $\mathbb{A}_0 = 1$, is

$$\mathbb{A}(t)e^{xt} = e^{\mathbb{A}.}e^{xt} = e^{(\mathbb{A}.+x)t} = e^{\mathbb{A},(x)t}.$$

The lowering op L defined by

$$L\ \mathbb{A}_n(x) = n\ \mathbb{A}_{n-1}(x)$$

is

$$L = D_x$$

since

$$D_x\mathbb{A}_n(x) = D_x\ (\mathbb{A}. + x)^n = n\ (\mathbb{A}. + x)^{n-1} = n\ \mathbb{A}_{n-1}(x).$$

The raising op R defined by

$$R\ \mathbb{A}_n(x) = \mathbb{A}_{n+1}(x)$$

is

$$R = \mathbb{A}(D_x)\ x\ \frac{1}{\mathbb{A}(D_x)} = x + D_{t=D_x} \ln[\mathbb{A}(t)].$$

An understanding of umbral compositional inversion in operator form is central to understanding the raising op.

First note the shift diff op action

$$\mathbb{A}(D_x)x^n = e^{\mathbb{A}.D_x}x^n = (\mathbb{A}. + x)^n = \mathbb{A}_n(x)$$

and define the umbral inverse Appell sequence $\hat{\mathbb{A}}_n(x)$ by the reciprocal action

$$\frac{1}{\mathbb{A}(D_x)}x^n = \hat{\mathbb{A}}(D_x)x^n = (\hat{\mathbb{A}}. + x)^n = \hat{\mathbb{A}}_n(x)$$.

Then

$$x^n = \mathbb{A}(D_x)\frac{1}{\mathbb{A}(D_x)}x^n = \mathbb{A}(D_x)\hat{\mathbb{A}}(D_x)x^n$$

$$= \mathbb{A}(D_x)(\hat{\mathbb{A}} + x)^n = \mathbb{A}(D_x)\hat{\mathbb{A}}_n(x) = \hat{\mathbb{A}}_n(\mathbb{A} + x) = \hat{\mathbb{A}}_n(\mathbb{A}.(x)).$$

Conversely,

$$\mathbb{A}_n(\hat{\mathbb{A}}.(x)) = x^n.$$

Reprising, we have the umbral inverse pair of Appell sequences characterized by

$$\hat{\mathbb{A}}_n(\mathbb{A}.(x)) = \mathbb{A}_n(\hat{\mathbb{A}}.(x)) = (\hat{\mathbb{A}}. + \mathbb{A}. + x)^n = x^n.$$

Evaluating at $x = 0$ gives

$$(\hat{\mathbb{A}}. + \mathbb{A}.)^n = 0^n = \delta_n,$$

or, equivalently,

$$\mathbb{A}(t)\hat{\mathbb{A}}(t) = e^{\mathbb{A}.t}e^{\hat{\mathbb{A}}.t} = e^{(\mathbb{A}.+\hat{\mathbb{A}}.)t} = 1.$$

This also translates into the language of modified lower triangular Pascal matrices whose rows are the coefficients of the Appel polynomials with the main diagonal all ones. The two matrices corresponding to the Appell umbral inverse pair are a matrix inverse pair. The raising and lowering op can also be represented as matrices of infinite rank but not triangular.

Note $A^{(2)}(t)$ and $N^{(-2)}(t)$ are o.g.f.s rather than e.g.f.s, so the Appell moments for these two are $\mathbb{A}_k = k!A_k^{(2)}$ and $\mathbb{A}_k = k!N_k^{(-2)}$.
