Appell-Sheffer and Symmetric Polynomials

Tom Copeland, May 7, 2023, Los Angeles

The generating function for a generic Appell polynomial sequence $\mathbb{A}_n(x)$ expressed first as an e.g.f. and then as an o.g.f. with moments $a_k = k!\bar{a}_k$ is

$$\mathbb{A}(t) = e^{a.t} = 1 + a_1 t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} + \cdots$$

$$= \bar{\mathbb{A}}(t) = 1 + \bar{a}_1 t + \bar{a}_2 t^2 + \bar{a}_3 t^3 + \cdot.$$

The differential part of the raising op $\mathbb{R}$ for an Appell Sheffer polynomial sequence, defined by

$\mathbb{R}\ \mathbb{A}_n(x) = \mathbb{A}_{n+1}(x)$,

is given via the logarithmic derivative of the Appell generating function

$$D_t \ln(\mathbb{A}(t)) = a_1 + (a_2 - a_1^2)t + (2a_1^3 - 3a_1 a_2 + a_3)\frac{t^2}{2!}$$

$$+(-6a_1^4 + 12a_1^2 a_2 - 3a_2^2 - 4a_1 a_3 + a_4)\frac{t^3}{3!}$$

$$+(24a_1^5 - 60a_1^3 a_2 + 20a_1^2 a_3 - 10a_2 a_3 + 30a_1 a_2^2 - 5a_1 a_4 + a_5)\frac{t^4}{4!} + \cdots$$

$$= D_t \ln(\bar{\mathbb{A}}(t)) = \bar{a}_1 + (2\bar{a}_2 - \bar{a}_1^2)t + (\bar{a}_1^3 - 3\bar{a}_2\bar{a}_1 + 3\bar{a}_3)t^2$$

$$+(-\bar{a}_1^4 + 4\bar{a}_2\bar{a}_1^2 - 4\bar{a}_3\bar{a}_1 - 2\bar{a}_2^2/2 + 4\bar{a}_4)t^3$$

$$+(\bar{a}_1^5 - 5\bar{a}_2\bar{a}_1^3 + 5\bar{a}_3\bar{a}_1^2 + 5\bar{a}_1\bar{a}_2^2 - \bar{a}_1\bar{a}_4 - 5\bar{a}_2\bar{a}_3 + 5\bar{a}_5)t^5 + \cdots$$

with the first set of ParPs being the logarithmic ParPs $L_n(a_1, ..., a_n)$_ of A263634, or cumulant expansion polynomials of 127671, and the second, the negative of the Faber polynomials $F_n(\bar{a}_1, ..., \bar{a}_n)$ of A263916.

With the sign convention of the Wikipedia article on the Newton identities, the Faber polynomials give the formal power sums p_n in terms of the elementary symmetric polynomials / functions as, e.g.,

$p_1 = \bar{a}_1 = -F_1(\bar{a}_1) = L_1(a_1)$,

$-p_2 = 2\bar{a}_2 - \bar{a}_1^2 = -F_2(\bar{a}_1, \bar{a}_2) = L_2(a_1, a_2)$,

$p_3 = \bar{a}_1^3 - 3\bar{a}_2\bar{a}_1 + 3\bar{a}_3 = -F_3(\bar{a}_1, \bar{a}_2, \bar{a}_3) = L_3(a_1, a_2, a_3)/2!$,

$-p_4 = -\bar{a}_1^4 + 4\bar{a}_2\bar{a}_1^2 - 4\bar{a}_3\bar{a}_1 - 2\bar{a}_2^2/2 + 4\bar{a}_4 = -F_4(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4) = L_4(a_1, a_2, a_3, a_4)/3!$,

and, in general,

$p_n = (-1)^n F_n(\bar{a}_1, ..., \bar{a}_n) = (-1)^{n+1} L_n(a_1, ..., a_n)/n!$.

Then the Appell raising op can be expressed as the diff op

$$\mathbb{R} = x + L_1(a_1) + L_2(a_1, a_2)D_x + L_3(a_1, a_2, a_3)\frac{D_x^2}{2!} + \cdots$$

$$= x - F_1(\bar{a}_1) - F_2(\bar{a}_1, \bar{a}_2)D_x - F_3(\bar{a}_1, \bar{a}_2, \bar{a}_3)D_x^2 + \cdots$$

$$= x + p_1 - p_2 D_x + p_3 D_x^2 - p_4 D_x^3 + \cdots$$

Action on the Appell polynomials gives

$$\mathbb{R}\ (a. + x)^n = (a. + x)^{n+1}.$$

This identity, when evaluated at $x = 0$, becomes, with $b_k = (-1)^k k! p_{k+1}$,

$$\sum_{k=0}^{\infty} (-1)^k p_{k+1} D_{x=0}^k (a. + x)^n = a_{n+1}$$

$$= (a. + b.)^n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

$$= \sum_{k=0}^{n} n! \bar{a}_k (-1)^{n-k} p_{n-k+1},$$

so

$$(n+1)\bar{a}_{n+1} = \sum_{k=0}^{n} \bar{a}_k p_{n-k+1}.$$

For example,

$$3\bar{a}_3 = \sum_{k=0}^{2} (-1)^k \bar{a}_k p_{2-k+1} = p_3 - p_2\bar{a}_1 + p_1\bar{a}_2,$$

in agreement with the Wikipedia article.

The umbral inverse Appell sequence $\hat{\mathbb{A}}_n(x)$ then plays the role of the e.g.f. counterpart to the complete homogeneous symmetric polynomials.

In previous posts and in several OEIS entries, e.g., A133314, I've illustrated the matrix formulation for the Appell calculus. The differential raising operation can be expressed in the matrix form in the power basis x^n (see, e.g., A094587, A099174, A039683, A130757, A111593, A176230, A111062, and A159834) as

$[\mathbb{R}][\mathbb{A}_1]$

$$= \begin{pmatrix} L_1 & L_2 & L_3 & L_4 & L_5 \\ 1 & L_1 & 2L_2 & 3L_3 & 4L_4 \\ 0 & 1 & L_1 & 3L_2 & 6L_3 \\ 0 & 0 & 1 & L_1 & 4L_2 \\ 0 & 0 & 0 & 1 & L_1 \end{pmatrix} \begin{pmatrix} a_1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_1 a_1 + L_2 \\ a_1 + L_1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_2 \\ 2a_1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Spot check:

$$\mathbb{R}\ \mathbb{A}_1 = \mathbb{R}\ (x+a_1) = (x + L_1(a_1) + L_2(a_1,a_2)D_x)(x+a_1)$$

$$= (x+L_1)(x+a_1) + L_2 = (L_2 + L_1 a_1) + (L_1 + a_1)x + x^2$$

$$= ((a_2 - a_1^2) + a_1 a_1) + (a_1 + a_1)x + x^2 = a_2 + 2a_1 x + x^2 = (a.+x)^2 = \mathbb{A}_2(x).$$

Next,

$[\mathbb{R}][\mathbb{A}_1]$

$$= \begin{pmatrix} L_1 & L_2 & L_3 & L_4 & L_5 \\ 1 & L_1 & 2L_2 & 3L_3 & 4L_4 \\ 0 & 1 & L_1 & 3L_2 & 6L_3 \\ 0 & 0 & 1 & L_1 & 4L_2 \\ 0 & 0 & 0 & 1 & L_1 \end{pmatrix} \begin{pmatrix} a_2 \\ 2a_1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_1 a_2 + 2L_2 a_1 + L_3 \\ a_2 + 2L_1 a_1 + 2L_2 \\ 2a_1 + L_1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_3 \\ 3a_2 \\ 3a_1 \\ 1 \\ 0 \end{pmatrix}.$$

Check:

$$\mathbb{R}\ \mathbb{A}_2 = R\ (a_2 + 2a_1 x + x^2) = (x + L_1 + L_2 D_x + L_3 \frac{D_x^2}{2!})(a_2 + 2a_1 x + x^2)$$

$$= (x + L_1)(a_2 + 2a_1 x + x^2) + L_2(2a_1 + 2x) + L_3$$

$$= (2a_1 L_2 + a_2 L_1 + L_3) + (2a_1 L_1 + a_2 + 2L_2)x + (2a_1 + L_1)x^2 + x^3.$$

$$2a_1 L_2 + a_2 L_1 + L_3 = 2a_1(a_2 - a_1^2) + a_2 a_1 + (2a_1^3 - 3a_1 a_2 + a_3) = a_3.$$

$$2a_1 L_1 + a_2 + 2L_2 = 2a_1 a_1 + a_2 + 2(a_2 - a_1^2) = 3a_2.$$

$$2a_1 + L_1 = 2a_1 + a_1 = 3a_1.$$

So, the raising calculation gives the coefficients of $\mathbb{A}_3(x)$.

Use $p_n = (-1)^n F_n(\bar{a}_1, ..., \bar{a}_n) = (-1)^{n+1} L_n(a_1, ..., a_n)/n!$ to replace L_n by $(-1)^{n+1} n! p_n$ and replace a_n by $n! \bar{a}_n = n! e_n$. Then

$$\begin{pmatrix} p_1 & -p_2 & 2p_3 & -6p_4 & 24p_5 \\ 1 & p_1 & -2p_2 & 6p_3 & -24p_4 \\ 0 & 1 & p_1 & -3p_2 & 12p_3 \\ 0 & 0 & 1 & p_1 & -4p_2 \\ 0 & 0 & 0 & 1 & p_1 \end{pmatrix} \begin{pmatrix} a_2 = 2e_2 \\ 2a_1 = 2e_1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_3 = 6e_3 \\ 3a_2 = 6e_2 \\ 3a_1 = 3e_1 \\ 1 \\ 0 \end{pmatrix}$$

Appell sequences in terms of $k! \bar{a}_k = a_k$ have the coefficients of A094587:

$$\mathbb{A}_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k} = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \bar{a}_k x^{n-k}$$
.

(See also A008279, A036039, A001497, and A133932 for significance of A094587.)

Let PM be the lower triangular permutation matrix of A094587 with components $P_{n,k} = \frac{n!}{k!}$ and $PM(1, e_1, e_2, e_3, ...)$ be the same matrix but with the j-th diagonal multiplied by e_j with $e_0 = 1$. Let APM be PM augmented with a single upper diagonal of ones.

Expressing the raising operator $\mathbb{R}$ for a generic Appell Sheffer polynomial sequence in terms of the power sums gives the matrix identity

$$PM(1, e_1, e_2, e_3, ...)APM(p_1, -p_2, p_3, -p_4, ...) = SPM(1, e_1, e_2, e_3, ...),$$

where the last matrix is a shifted variant of $PM(1, e_1, e_2, e_3, ...)$; i.e., SPM is $PM(1, e_1, e_2, e_3, ...)$ with the first column removed.

For example, the 5 by 5 truncated matrix version of this identity transposed to match the form of the Appell raising operation acting on a generic sequence of Appell polynomials is

$$APM_5^T \cdot PM_5^T = SPM_5^T$$

$$= \begin{pmatrix} p_1 & -p_2 & 2p_3 & -6p_4 & 24p_5 \\ 1 & p_1 & -2p_2 & 6p_3 & -24p_4 \\ 0 & 1 & p_1 & -3p_2 & 12p_3 \\ 0 & 0 & 1 & p_1 & -4p_2 \\ 0 & 0 & 0 & 1 & p_1 \end{pmatrix} \begin{pmatrix} 1 & e_1 & 2e_2 & 6e_3 & 24e_4 \\ 0 & 1 & 2e_1 & 6e_2 & 24e_3 \\ 0 & 0 & 1 & 3e_1 & 12e_2 \\ 0 & 0 & 0 & 1 & 4e_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e_1 & 2e_2 & 6e_3 & 24e_4 & 120e_5 \\ 1 & 2e_1 & 6e_2 & 24e_3 & 120e_4 \\ 0 & 1 & 3e_1 & 12e_2 & 60e_3 \\ 0 & 0 & 1 & 4e_1 & 20e_2 \\ 0 & 0 & 0 & 1 & 5e_1 \end{pmatrix}$$

or, with $a_k = k!e_k$,

$$\begin{pmatrix} p_1 & -p_2 & 2p_3 & -6p_4 & 24p_5 \\ 1 & p_1 & -2p_2 & 6p_3 & -24p_4 \\ 0 & 1 & p_1 & -3p_2 & 12p_3 \\ 0 & 0 & 1 & p_1 & -4p_2 \\ 0 & 0 & 0 & 1 & p_1 \end{pmatrix} \begin{pmatrix} 1 & a_1 & a_2 & a_3 & a_4 \\ 0 & 1 & 2a_1 & 3a_2 & 4a_3 \\ 0 & 0 & 1 & 3a_1 & 6a_2 \\ 0 & 0 & 0 & 1 & 4a_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 1 & 2a_1 & 3a_2 & 4a_3 & 5a_4 \\ 0 & 1 & 3a_1 & 6a_2 & 10a_3 \\ 0 & 0 & 1 & 4a_1 & 10a_2 \\ 0 & 0 & 0 & 1 & 5a_1 \end{pmatrix}.$$

Note that PM^T expressed in the indeterminates a_n is the transpose of the lower triangular Pascal matrix A007318 with the diagonal multiplication by the a_n's, so it is the transpose of the lower triangular coefficient matrix for the Appell polynomials

$A_n(x) = (a. + x)^n = \sum_{k=0}^{n} \binom{n}{k} a_{n-k} x^k$. Note also the induced determinant relation $|APM_5| = |SPM_5|$.

The matrix rep for the Appell raising op can be recast as the transpose of a Riordan production matrix. Perhaps the abundant literature on Riordan matrix theory contains the matrix identity and combinatorial proofs.

See the OEIS entry on Faber polynomials for more on their relationships to symmetric function theory, They are core constructs in number theory, complex analysis, and combinatorics. Gould in “The Girard-Waring power sum formulas for symmetric functions and Fibonnaci sequences” for some history and multinomial coefficients for the expansions of the power sums in terms of the elementary symmetric functions. See also "A Matrix Proof of Newton's Identities" by Dan Kalman for other proofs of some Newton identities.

Added May 9, 2023:

In my notes "The Creation / Raising Operators for Appell Sequences" and "Lagrange à la Lah Part I", I cast Appell polynomials in terms of three sets of compositional partition polynomials. (Recursion relations are available in the first pdf.)

The first set of compositional partition polynomials is the set of cycle index partition polynomials of the symmetric groups S_n, a.k.a. the refined Stirling partition polynomials of the first kind $ST1_n(b_1, ..., b_N)$ of A036039, an Appell sequence in the distinguished indeterminate b_1 with the raising operator

$$\mathbb{R}_{\mathbb{ST1}} = \frac{b.}{1 + b.D_{b_1}} = b_1 + \sum_{n \geq 1} b_{n+1} D_{b_1}^n$$

and the lowering operator

$$\mathbb{L}_{ST1} = D_{b_1} = \frac{\partial}{\partial b_1}.$$

The Appell polynomials above can be expressed as

$$\mathbb{A}_n(x) = (a. + x)^n = ST1_n(x + b_1, b_2, b_3, ..., b_n)$$

Since

$$\frac{\partial}{\partial x}\mathbb{A}_n(x) = \frac{\partial}{\partial b_1} ST1_n(x+b_1, b_2, b_3, ..., b_n),$$

the raising op of $\mathbb{A}_n(x)$ is

$$\mathbb{R}_{\mathbb{A}} = x + b_1 + \sum_{n\geq 1} b_{n+1} D_x^n.$$

Comparing with

$$\mathbb{R}_{\mathbb{A}} = x + p_1 - p_2 D_x + p_3 D_x^2 - p_4 D_x^3 + \cdots,$$

then

$$b_n = (-1)^{n+1} p_n$$

and

$$\mathbb{A}_n(0) = a_n = n! e_n = ST1_n(b_1, b_2, b_3, ..., b_n) = ST1_n(p_1, -p_2, b_3, ..., (-1)^{n+1} p_n).$$

Spot check:

$$ST1_3(u_1, u_2, u_3) = 2u_3 + 3u_1 u_2 + u_1^3$$

and

$$a_3 = 3! e_3 = ST1_3(p_1, -p_2, p_3) = 2p_3 + 3p_1(-p_2) + p_1^3$$

$$= 2(\bar{a}_1^3 - 3\bar{a}_2\bar{a}_1 + 3\bar{a}_3) + 3\bar{a}_1(2\bar{a}_2 - \bar{a}_1^2) + \bar{a}_1^3$$

$$= 3!\bar{a}_3.$$

The second is the set of compositional Faa di Bruno / Bell partition polynomials of A036040, a.k.a. the refined Stirling partition polynomials of the second kind $ST2_n(c_1, ..., c_n)$, an Appell sequence in the distinguished indeterminate c_1 with the raising operator

$$\mathbb{R}_{\mathbb{ST}\not{2}} = c.e^{c.D_{c_1}} = c_1 + \sum_{n\geq 1} \frac{c_{n+1}}{n!} D_{c_1}^n$$

and the lowering operator

$$\mathbb{L}_{ST2} = D_{c_1} = \frac{\partial}{\partial c_1}.$$

The Appell polynomials above can be expressed as

$\mathbb{A}_n(x) = (a. + x)^n = ST2_n(x + c_1, c_2, c_3, ..., c_n)$,

and the raising op of $\mathbb{A}_n(x)$, as

$$\mathbb{R}_{\mathbb{A}} = x + c_1 + \sum_{n\geq 1} \frac{c_{n+1}}{n!} D_x^n$$

through which we can identify

$$c_n = (-1)^{n+1}(n-1)!p_n$$

and

$\mathbb{A}_n(0) = a_n = n!e_n = ST2_n(p_1, -p_2, 2!p_3, -3!p_4, ..., (-1)^{n+1}(n-1)!p_n)$.

Spot check:

$$ST2_3(u_1, u_2, u_3) = u_3 + 3u_1u_2 + u_1^3$$

and

$a_3 = 3!e_3 = ST2_3(2!p_1, -3!p_2, ..., (-1)^{n+1}(n+1)!p_n) = 2!p_3 + 3(p_1)(-p_2) + (p_1)^3$

$= 2!(\bar{a}_1^3 - 3\bar{a}_2\bar{a}_1 + 3\bar{a}_3) + 3(\bar{a}_1)(2\bar{a}_2 - \bar{a}_1^2) + (\bar{a}_1)^3$

$= 3!\bar{a}_3.$

The third is the set of the refined Lah partition polynomials $Lah_n(d_1, ..., d_n)$ of A130561, an Appell sequence in the distinguished indeterminate d_1 with the raising operator

$$\mathbb{R}_{\mathbb{L}\eth\approx} = \frac{d.}{(1-d.D_{d_1})^2} = d_1 + \sum_{n\geq 1}(n+1)d_{n+1}D_{d_1}^n$$

and the lowering operator

$$\mathbb{L}_{Lah} = D_{d_1} = \frac{\partial}{\partial d_1}.$$

The Appell polynomials above can be expressed as

$$\mathbb{A}_n(x) = (a. + x)^n = Lah_n(x + d_1, d_2, d_3, ..., d_n),$$

and the raising op of $\mathbb{A}_n(x)$, as

$$\mathbb{R}_{\mathbb{A}} = x + d_1 + \sum_{n\geq 1}(n+1)d_{n+1}D_x^n$$

through which we can identify

$$d_n = (-1)^{n+1}\frac{p_n}{n}$$

and

$$\mathbb{A}_n(0) = a_n = n!e_n = Lah_n(p_1, -b_2/2, b_3/3, ..., (-1)^{n+1}p_n/n).$$

Spot check:

$$Lah_3 = 6u_3 + 6u_1u_2 + u_1^3$$

and

$$a_3 = 3!e_3 = Lah_3(p_1, -p_2/2, b_3/3) = 6(p_3/3) + 6(p_1)(-p_2/2) + (p_1)^3$$

$$= 6((\bar{a}_1^3 - 3\bar{a}_2\bar{a}_1 + 3\bar{a}_3)/3) + 6(\bar{a}_1)((2\bar{a}_2 - \bar{a}_1^2)/2) + (\bar{a}_1)^3$$

$$= 3!\bar{a}_3.$$