Generalized Schur expansion Koefficients

K_n^z

Convolutions, compositions, and derivatives of generalized Schur coefficients in concert with the Faber, refined Lah, and refined and coarse Stirling polynomials

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As discussed in my pdf “One matrix to rule them all”, Schur, in his posthumously published paper "Identities in the theory of power series", presented the Schur generalized iterated self-convolution expansion coefficients, or, more briefly, the Schur expansion Koefficients, K_n^p defined as the coefficients of the power series expansion of the integer power p of a formal power series, or ordinary generating function:

the Schur expansion Koefficients

$$(1+h_1x+h_2x^2+\cdots)^p = (h(x))^p = \sum_{k\geq 0} K_k^p(h_1,\cdots,h_k)\; x^k.$$

I use the term self-convolution in a generalized sense--strictly speaking, the phrase 'iterated self-convolution' can be applied only when p is a natural number, i.e., $0,1,2,3,\cdots$, for which the Schur coefficients can be derived as iterated Cauchy convolutions; however, the definition above is sensible and useful for p any integer, i.e., $0,\pm1,\pm2,\pm3,\cdots$ and, as we shall see below, for any complex number.

Taking the logarithm of the definition provides links to the classic Faber partition polynomials F_n of A263916, found among the Newton-Waring-Girard identities for symmetric polynomials and defined as

$$-\ln(h(x)) = -\ln(1+h_x+h_2x^2+\ldots) = \sum_{m\geq 1} \frac{F_m(h_1,\cdots,h_m)}{m} x^m$$

$$= -h_1 + (-2h_2+h_1^2)x + (-h_1^3+3h_2h_1-3h_3)x^2 + (h_1^4-4h_2h_1^2+4h_3h_1+2h_2^2-4h_4)x^3$$

$$+\,(-h_1^5+5h_2h_1^3-5h_3h_1^2-5h_2^2h_1+5h_4h_1+5h_2h_3-5h_5)\;x^4+\cdots.$$

Then

$\ln[(h(x))^n/n] = \ln(n) + n\ln(h(x)),$

so

$D_x \ln[(h(x))^n/n] = n\frac{h'(x)}{h(x)}$

and

$D_x \frac{(h(x))^n}{n} = (h(x))^{n-1}h'(x) = n\frac{(h(x))^n}{n}\frac{h'(x)}{h(x)} = (h(x))^n D_x \ln(h(x))$

establishes

$n\frac{h'(x)}{h(x)}(h(x))^n = -n\ (\sum_{m\geq 0} F_{m+1}x^m) \sum_{k\geq 0} K_k^n x^k$

$= -n\ \sum_{m\geq 0}[\sum_{k=0}^m F_{k+1}K_{m-k}^n]x^m$

$= D_x(h(x))^n = \sum_{m\geq 0}(m+1)\ K_{m+1}^n x^m,$

from which we can glean

$(m+1)\ K_{m+1}^n = -n\sum_{k=0}^m F_{k+1}K_{m-k}^n,$

or,

the convolutional Faber-Schur partition polynomial recursion relation

$K_{m+1}^n = -\frac{n}{m+1}\sum_{k=0}^m F_{k+1}K_{m-k}^n$.

As discussed below, the superscript n of the Schur expansion coefficients can be any complex number.

(In “Lattice paths and Faber polynomials", Gessel and Ree use the notation $F_n(u)$ for what they call the Faber polynomials of univalent function theory, related to the Faber partition polynomials by $F_n(h_1, \ldots,_n) = F_n(0)$ At the bottom of page 4 of G & R is a multinomial-type coefficient for the numerical factors of their $F_n(u)$.)

Spot check:

$F_1(c_1) = -c_1$,
$F_2(c_1, c_2) = -2c_2 + c_1^2$,
$F_3(c_1, c_2, c_3) = -3c_3 + 3c_1c_2 - c_1^3.$

The relevant Schur expansion coefficients are determined from

$$(1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4)^n$$

$$= 1 + c_1nx + x^2(1/2c_1^2(n-1)n + c_2n) + x^3(1/6c_1^3(n-2)(n-1)n + c_2c_1(n-1)n + c_3n) + \cdots ,$$

giving

$K_0^n = 1$,
$K_1^n = n\ c_1$,
$K_2^n = \frac{1}{2}(n-1)\ n\ c_1^2 + n\ c_2$,
$K_3^n = \frac{1}{6}\ (n-2)(n-1)n\ c_1^3 + (n-1)n\ c_2c_1(n-1)n + n\ c_3.$

Then, from the general identity

$$K_{m+1}^n = -\tfrac{n}{m+1} \textstyle\sum_{k=0}^m F_{k+1} K_{m-k}^n,$$

we have the corroborating spot check

$$K_2^n = -\tfrac{n}{2} \textstyle\sum_{k=0}^1 F_{k+1} K_{1-k}^n = -n[F_1K_1^n + F_2K_0^n]$$

$$= -\tfrac{1}{2}n[(-c_1)(nc_1) + (-2c_2 + c_1^2)] = \tfrac{1}{2}\ (n-1)\ n\ c_1^2 + n\ c_2\ .$$

The generalized Schur expansion coefficients and the refined Lah polynomials

Those with the habit of using the OEIS and normalizing polynomials to obtain integer coefficients would find that the expansion coefficients are specializations of the elementary Schur polynomials, a.k.a. the refined Lah composition partition polynomials of A130561, one of my favorites not only because of its connections to symmetric function theory but also because it's obvious reductions give the Lah polynomials of A105278 and A111596, related to the Laguerre polynomials of order 1 and -1, the foundations for the sequences of orthogonal associated Laguerre polynomials, of which two sequences comprise the family of Hermite polynomials so important to harmonic analysis and classical and quantum physics. Indeed, the generalized Laguerre polynomials are the confluent hypergeometric functions, which encompass pretty much the classical analysis of vibrational physical phenomena, and A130561 has links I've provided on integrable hierarchies of the KdV and KP equation, various combinatorial constructs, and quantum physics. In addition, the associahedra o.g.f. inversion polynomials of A133437 can be umbrally couched in terms of these refined Lah polynomials, an Appell Sheffer sequence in the distinguished indeterminate a_1, and the Lah numbers. They are essentially the basic partition polynomials, i.e., coproduct, for a combinatorial Hopf algebra with the refined Euler characteristic polynomials of the associahedra as the antipode, that is the o.g.f. equivalent of the Faa di Bruno-Bell Hopf algebra for exponential generating functions (e.g.f.s) with the classic Lagrange inversion polynomials of A134685 as the antipode and the basic partition polynomials, i.e., coproduct, the Faa di Bruno-Bell composition polynomials of A036040.

The e.g.f. for ***the refined Lah polynomials*** is, in umbral notation with $(\widehat{Lah}.(t))^k = \widehat{Lah}_k(t) = Lah_k(a_1, \cdots, a_k; t)$,

$$e^{x \cdot \widehat{Lah}.(t)} = \exp[t \cdot (a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots)] = e^{tf(x)}$$

$$= 1 + a_1 \; t \; x + (a_1^2 t^2 + 2a_2 t)\frac{x^2}{2!} + (a_1^3 t^3 + 6a_2 a_1 t^2 + 6a_3 t)\frac{x^3}{3!}$$

$$+ (a_1^4 t^4 + 12a_2 a_1^2 t^3 + 12a_2^2 t^2 + 24a_3 a_1 t^2 + 24a_4 t)\frac{x^4}{4!}$$

$$+ (a_1^5 t^5 + 20a_2 a_1^3 t^4 + 60a_3 a_1^2 t^3 + 60a_1 a_2^2 t^3 + 120a_1 a_4 t^2 + 120a_2 a_3 t^2 + 120a_5 t)\frac{x^2}{2!} + \cdots$$

(note $Lah_k(a_1, \cdots, a_k; t) = Lah_k(a_1 t, \cdots, a_k t; 1)$)

and, for comparison,

$$(1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots)^n$$

$$= 1 + c_1 n x + [c_1^2 \tfrac{n!}{(n-2)!} + 2c_2 \tfrac{n!}{(n-1)!}]\frac{x^2}{2!} + [c_1^3 \tfrac{n!}{(n-3)!} + 6c_2 c_1 \tfrac{n!}{(n-2)!} + 6c_3 \tfrac{n!}{(n-1)!}]\frac{x^3}{3!}$$

$$+ [c_1^4 \tfrac{n!}{(n-4)!} + 12c_2 c_1^2 \tfrac{n!}{(n-3)!} + (12c_2^2 + 24c_1 c_3)\tfrac{n!}{(n-2)!} + 24c_4 \tfrac{n!}{(n-1)!}]\frac{x^4}{4!}$$

$$+ [c_1^5 \frac{n!}{(n-5)!} + 20c_2c_1^3 \frac{n!}{(n-4)!} + 60c_3c_1^2 \frac{n!}{(n-3)!} + \ 60c_1c_2^2 \frac{n!}{(n-3)!}$$

$$+120c_1c_4 \frac{n!}{(n-2)!} + 120c_2c_3 \frac{n!}{(n-2)!} \ + 120c_5 \frac{n!}{(n-1)!}] \frac{x^5}{5!} + \cdots$$

By inspection,

$$(h(x))^n = \sum_{k\geq 0} Lah_k(c_1, \cdots, c_k; St1.(n)]$$

where $St1_k(x) = \frac{x!}{(x-k)!} = x(x-1)\cdots(x-k+1)$ are the Stirling polynomials of the first kind, or the falling factorials, of A048993 (variants: A008275, A094638, refinement: A036039) and the umbral maneuver $(St1.(x))^k = St1_k(x)$ is performed after the umbral substitution of $St.(n)$ for t in $Lah_k(c_1, \cdots; t)$. This was done in my notes “Lagrange a la Lah” to obtain an umbral expression for the associahedra compositional inversion polynomials of A133437,which are $\frac{K_{n-1}^n}{n-1}$.

This substitution can be represented with $u_n = St1_n(z) = \frac{z!}{(z-n+1)!}$ by

$$e^{u.D_{t=0}} \ e^{tf(x)} = e^{u.f(x)} = \sum_{n\geq 0} u_n \frac{(f(x))^n}{n!}$$

$$= \sum_{n\geq 0} \frac{z!}{n!}(f(x))^n = (1+f(x))^z = (h(x))^z = \sum_{n\geq 0} K_n^z x^n,$$

$$= e^{u.D_{t=0}} e^{x\widehat{Lah}.(t)} = e^{x\widehat{Lah}.(u.)} = \sum_{n\geq 0} \widehat{Lah}_n(u.)x^n,$$

giving the analytic continuation / interpolation of the superscript p of the Schur expansion coefficients from the integers to any complex number as

the generalized Schur expansion coefficients

$$K_n^z(a_1, \cdots, a_n) = \widehat{Lah}_n[St1.(z)]$$

$$= Lah_n(a_1, \cdots, a_n; St1.(z)] = Lah_n(a_1 St1.(z), \cdots, a_n St1.(z); 1].$$

The refined Lah polynomials are a binomial Sheffer sequence in x with a binary tree rep and raising / creation and lowering / annihilation / destruction, or ladder, ops and a Graves-Lie infinitesimal generator. In the distinguished indeterminate c_1, they are an Appell sequence with ladder ops defined in terms of derivatives and multiplication w.r.t. c_1. For more info on these fascinating partition polynomials, see my post "Lagrange a la Lah" and various others of mine as well as the OEIS entry. All these polynomials are rife with combinatorial interpretations and closely associated with the Coxeter groups A_n.

A quick perusal of A130561 will reveal that there are many identities for the refined Lah polynomials involving their derivatives w.r.t. to their indeterminates and the Faber partition polynomials as a means of extracting the values of indeterminates of the refined Lah polynomials given evaluations of the polynomials, i.e., inverting the partition polynomials at the indeterminate level. So, it's not surprising that the same is true of the Schur expansion coefficients since the Stirling polynomials of the second kind $St2_n(x)$, a.k.a. the Bell-Touchard-Steffensen-exponential polynomials, of A048993 (variant A008277, refinement: A036040) are the umbral inverse polynomials of the $St1_n(x)$ i.e., $St1_n[St2.(x)] = x^n = St2_n[St1.(x)]$, so we can easily convert between the generalized Schur expansion coefficients and the refined Lah polynomials, preserving many properties of the two sets.

(In conventional presentations, this section would be the first section of these notes, but that would hide the path of discovery, which to me is typically both a more interesting and logical way, in the sense of a more suggestive and easily remembered flow of ideas, to clothe a subject than the staid, bloodless definition-lemma-theory-proof-corollary straitjacket.)

An identity for the Faber polynomials as a convolution of the Schur expansion coefficients can similarly be derived:

$$(D_x \frac{(h(x))^n}{n})(h(x))^{-n} = \frac{h'(x)}{h(x)}$$

$$= (D_x \frac{(\sum_{k\geq 0} K_k^n \ x^k)}{n})(\sum_{k\geq 0} K_k^{-n} \ x^k)$$

$$= \sum_{m\geq 0} -F_{m+1}(c_1, ..., c_{m+1}) \ x^m,$$

$$= (\frac{(\sum_{k\geq 0}(k+1) \ K_{k+1}^n \ x^k)}{n})(\sum_{k\geq 0} K_k^{-n} \ x^k),$$

implying

another Schur-Faber convolution identity

$F_{m+1}(c_1, ..., c_{m+1}) = -\frac{1}{n} \sum_{k=0}^{m} (k+1)\ K_{k+1}^{n} K_{m-k}^{-n}.$

The superscript n of the Schur expansion coefficients can be any complex number.

Spot check:

$(1 + c_1 x + c_2 x^2 + c_3 x^3 + ...)^2 = 1 + 2c_1 x + (c_1^2 + 2c_2)x^2 + 2(c_1 c_2 + c_3)x^3 + \cdots$

and

$(1 + c_1 x + c_2 x^2 + c_3 x^3)^{-2} = 1 - 2c_1 x + (3c_1^2 - 2c_2)x^2 - 2(2c_1^3 - 3c_2 c_1 + c_3)x^3 + \cdots ,$

so

$\frac{1}{2} \sum_{k=0}^{2} (k+1)\ K_{k+1}^{2} K_{2-k}^{-2} = \frac{1}{2}[K_1^2 K_2^{-2} + 2K_2^2 K_1^{-2} + 3K_3^2 K_0^{-2}]$

$= \frac{1}{2}[(2c_1)(3c_1^2 - 2c_2) + 2(c_1^2 + 2c_2)(-2c_1) + 3(2(c_1 c_2 + c_3))(1)]$

$= c_1^3 - 3c_1 c_2 + 3c_3 = -F_3(c_1, c_2, c_3).$

Compositional identities also exists among the three sets of Faber and Schur coefficient polynomials and the refined Stirling partition polynomials of the first kind:

$\ln[(h(x))^n] = \ln[\sum_{n \geq 0} K_k^n x^k] = \sum_{m \geq 1} -\frac{F_m(K_1^n, ..., K_m^n)}{m} x^m$

$= n \ln[h(x)] = n \ln[h(x)] = n\ \ln(1 + c_x + c_2 x^2 + ...)$

$= n\ \sum_{m \geq 1} -\frac{F_m(c_1, ..., c_m)}{m} x^m,$

so we have the ***compositional identity***

$\frac{F_m(K_1^n, ..., K_m^n)}{n} =\ F_m(c_1, ..., c_m).$

Again, the superscript n of the Schur expansion coefficients can be any complex number.

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Using the refined Stirling polynomials of the first kind $RSt1_m$, a.k.a. the cycle index polynomials for symmetric groups S_n of A036039, the c_n and K_m^n can be extracted as indicated in A263916:

indeterminate extractions

$$RSt1_m(F_1(c_1), ..., F_m(c_1, ..., c_m)) = m!c_m = RSt1_m(\frac{F_1(K_1^n))}{n}, ..., \frac{F_m(K_1^n, ..., K_m^n)}{n})$$

and

$$RSt1_m(F_1(K_1^n)), \cdots, F_m(K_1^n, \cdots, K_m^n)) = m!K_m^n$$

$$= RSt1_m(n\ F_1(c_1), \cdots, n\ F_m(c_1, \cdots, c_m)).$$

The superscript n of the Schur expansion coefficients can be any complex number.

Derivatives with respect to indeterminates

Taking derivatives w.r.t. the indeterminates introduces a multitude of identities. A basic one, which I will use in a forthcoming set of notes to derive an umbral recursion formula for the special normalized Schur expansion coefficients $b_n = -\frac{K_n^{n-1}}{n-1}$, follows from

$$\sum_{k \geq 0} D_{h_m} K_k^n(h_1, \cdots, h_k) x^k = D_{h_m}(h(x))^n$$

$$= n\ (h(x))^{n-1} x^m = n\ \sum_{k \geq m} K_{k-m}^{n-1}(h_1, \cdots, h_{k-m}) x^k,$$

Implying the ***basic derivative identity***

$D_{c_m} K_k^n(c_1, \cdots, c_k) = 0$ for $0 \leq k < m$ and otherwise

$$D_{c_m} K_k^n(c_1, \cdots, c_k) = n\ K_{k-m}^{n-1}(c_1, \cdots, c_{k-m}).$$

Spot checks:

For $m = 2$, $n = 2$, and $k = 3$:

$$D_{c_2} K_3^2 = D_{c_2} 2(c_1 c_2 + c_3) = 2c_1 = 2K_1^1.$$

For $m = 2$, $n = 3$, and $k = 4$:

$$D_{c_2} K_4^3 = D_{c_2} 3(c_2 c_1^2 + 2c_3 c_1 + c_2^2 + c_4) = 3(c_1^2 + 2c_2) = 3K_2^2.$$

A relation to the general Bell-Faa di Bruno composition partition polynomials:

$$D_x (h(x))^n = n(h(x))^{n-1} h'(x) = n(h(x))^n D_x \ln(h(x));$$

second derivative gives

$$D_x^2 (h(x))^n = n\ h^n(x)\ [n\ (D_x \ln(h(x)))^2 + D_x^2 \ln(h(x))];$$

third,

$$D_x^3 (h(x))^n$$

$$= n\ (h(x))^n\ [n^2\ (D_x \ln(h(x)))^3 + 3n\ (D_x \ln(h(x)))(D_x^2 \ln(h(x)) + D_x^3 \ln(h(x))];$$

fourth,

$$D_x^4 (h(x))^n$$

$$= n\ (h(x))^n\ [n^3\ (D_x \ln(h(x)))^4 + 6n^2\ (D_x \ln(h(x)))^2 (D_x^2 \ln(h(x)))$$

$$+ 4n(D_x \ln(h(x))) D_x^3 \ln(h(x)) + 3n\ (D_x^2 \ln(h(x)))^2 + D_x^4 \ln(h(x))];$$

Compare these polynomials with those of A036040, the refined Stirling polynomials of the second kind, a.k.a. the general Faa di Bruno-Bell partition polynomials. with

$(h(x))^n = \exp[n \ln(h(x))] = \exp[tf(x)]|_{t=n}$.

Then

$\frac{d^4}{dx^4}\ \exp[tf(x)]$

$= te^{tf(x)}[f^(4)(x) + 3tf''(x)^2 + t^3 f'(x)^4 + 4tf^{(3)}(x)f'(x) + 6t^2 f'(x)^2 f''(x)]$

and

$\frac{d^5}{dx^5}\ \exp[tf(x)]$

$= te^{tf(x)}[f^{(5)}(x) + t^4 f'(x)^5 + 10tf^{(3)}(x)f''(x) + 10t^2 f^{(3)}(x)f'(x)^2$

$+10t^3 f'(x)^3 f''(x) + 5tf'(x)(f^{(4)}(x) + 3tf''(x)^2)]$.

The relation is obvious, but it’s the refined Lah polynomials that provide a more illuminating view.
