Local compositional inversion of f(x) = x / (1 + b x + c x^2)

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The local compositional inversion of the curve $f(x) = \frac{x}{1+bx+cx^2}$ is portrayed by reflection through the quadrant bisector $y = x$ to blue and green segments of the blue and green curves with

$f(x) = \frac{x}{1+bx+cx^2}$ red curve,

$g(x) = \frac{1-bx-\sqrt{(1-bx)^2-4cx^2}}{2cx}$ blue curve,

$h(x) = \frac{1-bx+\sqrt{(1-bx)^2-4cx^2}}{2cx}$ green curve,

and

$y = x$ black bisecting line.

Series expansions about $x = 0$:

$f(x) = \frac{x}{1+ax+cx^2}$

$= x + (-b)\ x^2 + (b^2 - c)\ x^3 + (2bc - b^3)\ x^4 + (b^4 - 3b^2c + c^2)\ x^5$

$+(-b^5 + 4b^3c - 3bc^2)\ x^6 + (b^6 - 5b^4c + 6b^2c^2 - c^3)\ x^7$

$+(-b^7 + 6b^5c - 10b^3c^2 + 4bc^3)\ x^8 + (b^8 - 7b^6c + 15b^4c^2 - 10b^2c^3 + c^4)\ x^9 + \cdots$

See OEIS A049310 as primary entry for this series, but also see A011973, A092865, A098925, A102426, and A169803.

Series in x of

$g(x) = \frac{1-bx-\sqrt{(1-bx)^2-4cx^2}}{2cx}$

$= x + (b)\ x^2 + (b^2 + c)\ x^3 + (b^3 + 3bc)\ x^4 + (b^4 + 6b^2c + 2c^2)\ x^5$

$+(b^5 + 10b^3c + 10bc^2)\ x^6 + (b^6 + 15b^4c + 30b^2c^2 + 5c^3)\ x^7$

$+(b^7 + 21b^5c + 70b^3c^2 + 35bc^3)\ x^8 + (b^8 + 28b^6c + 140b^4c^2 + 140b^2c^3 + 14c^4)\ x^9 + \cdots$

See A097610 for these coefficients.

Lorentz series in x of

$h(x) = \frac{1-bx+\sqrt{(1-bx)^2-4cx^2}}{2cx}$

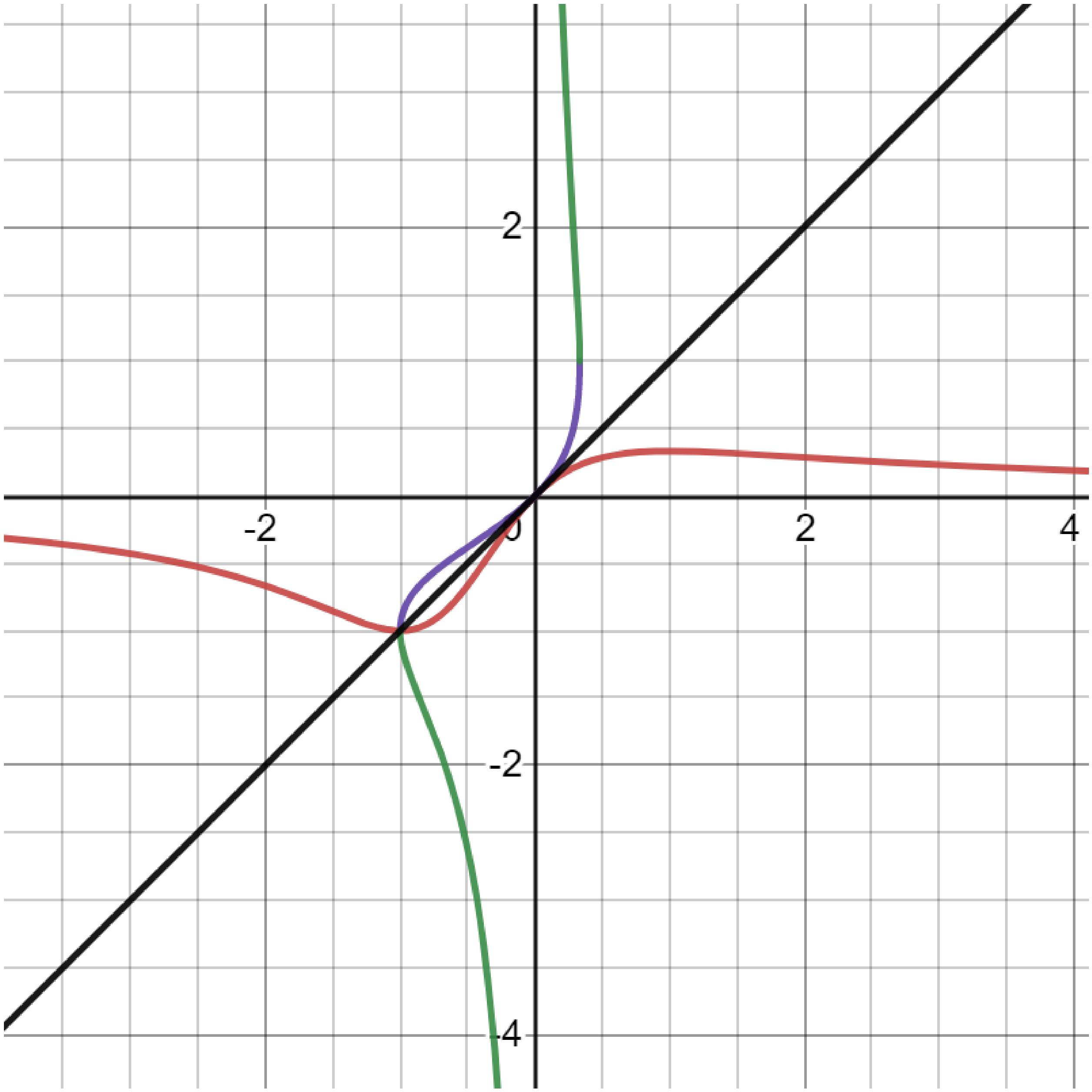
$= \frac{1}{cx} - \frac{b}{c} - (1)x - (b)x^2 - (b^2 + c)x^3 - (b^3 + 3bc)\ x^4 - (b^4 + 6b^2c + 2c^2)\ x^5$

$-(b^5 + 10b^3c + 10bc^2)\ x^6 - (b^6 + 15b^4c + 30b^2c^2 + 5c^3)\ x^7 + \cdots$

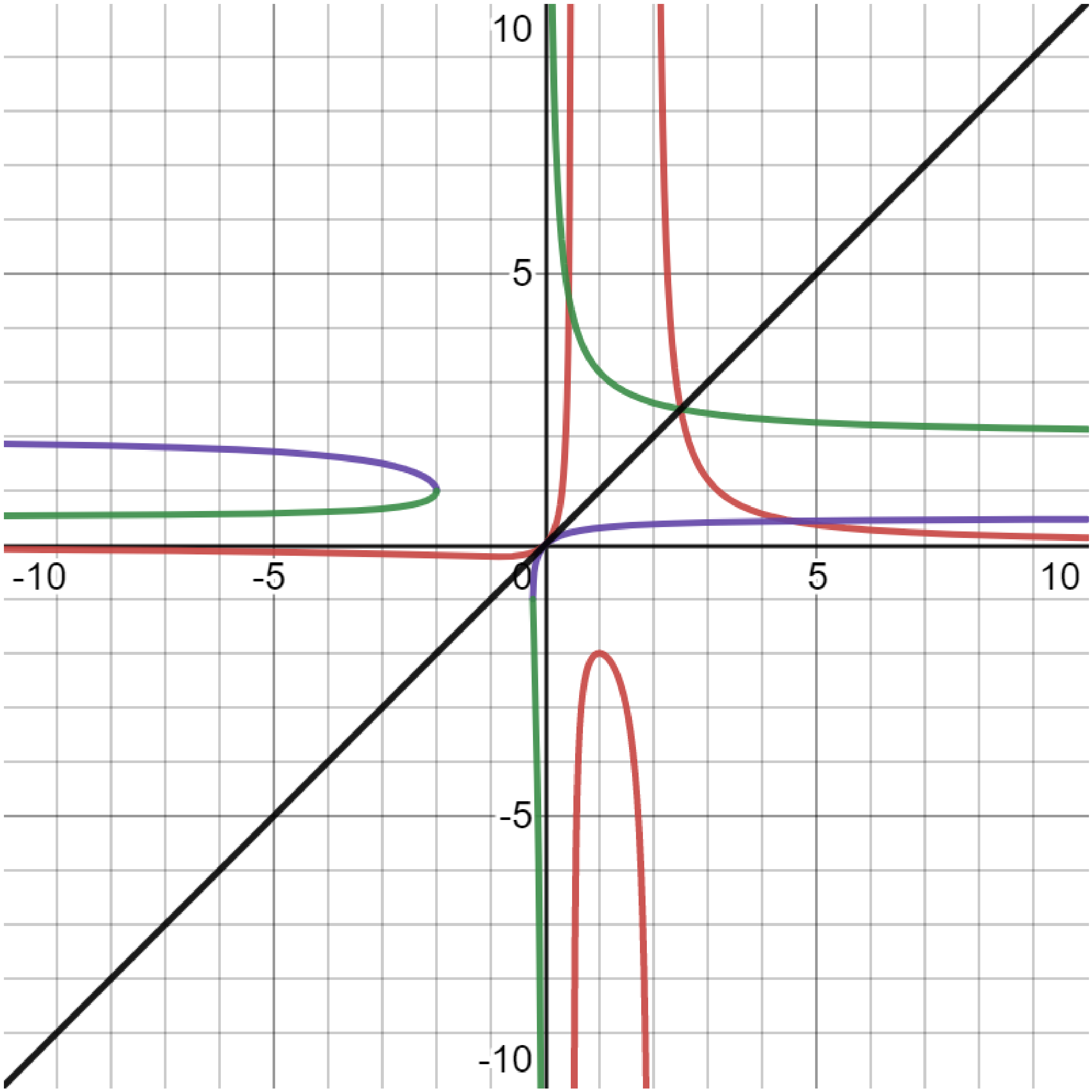
$= \frac{1-bx}{cx} - g(x).$

(Compare this with “Appendix: Laurent series formulation” of the pdf “Ruling the inverse universe, the inviscid Hopf-Burgers evolution equation”.)

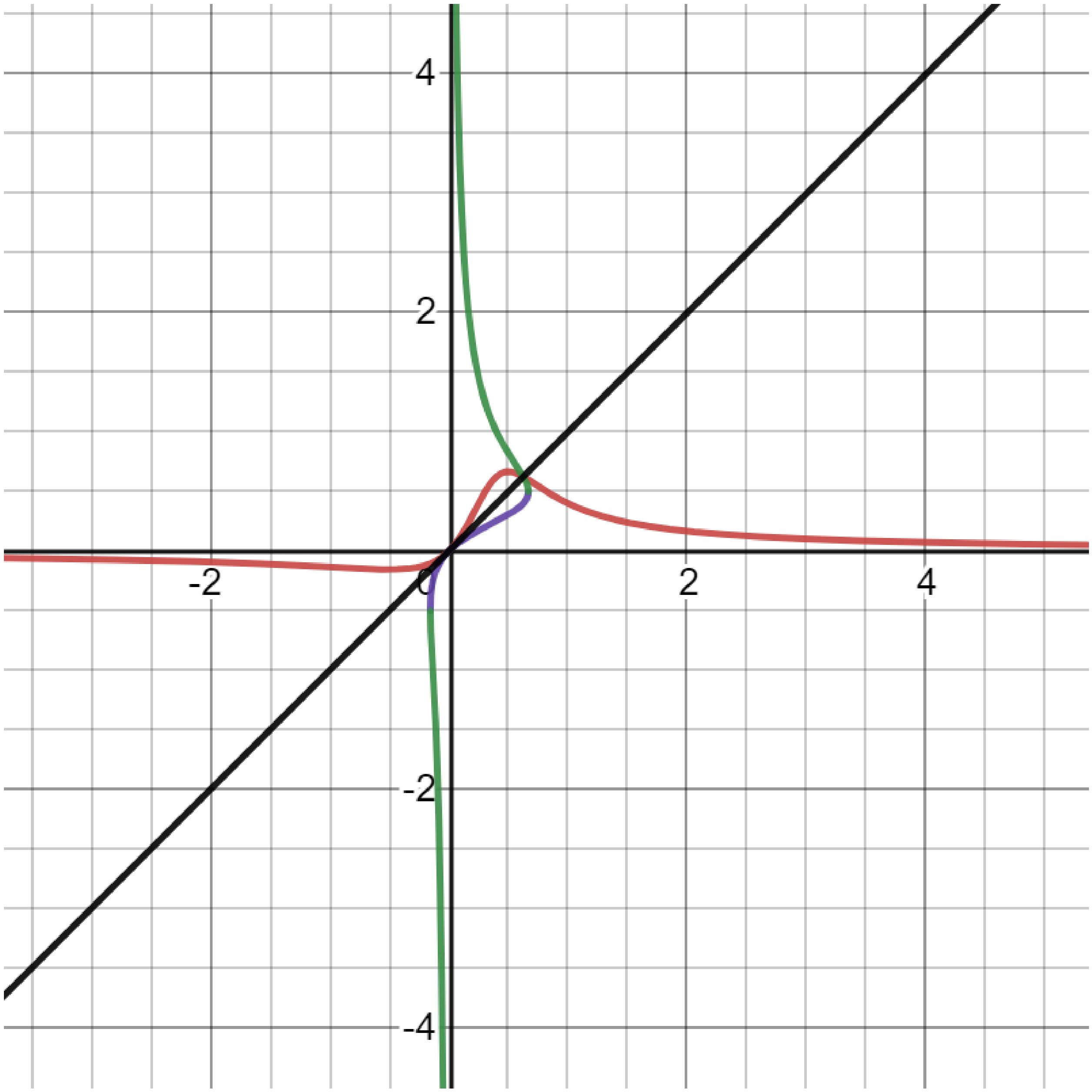
Graphs via Desmos online graphing app:



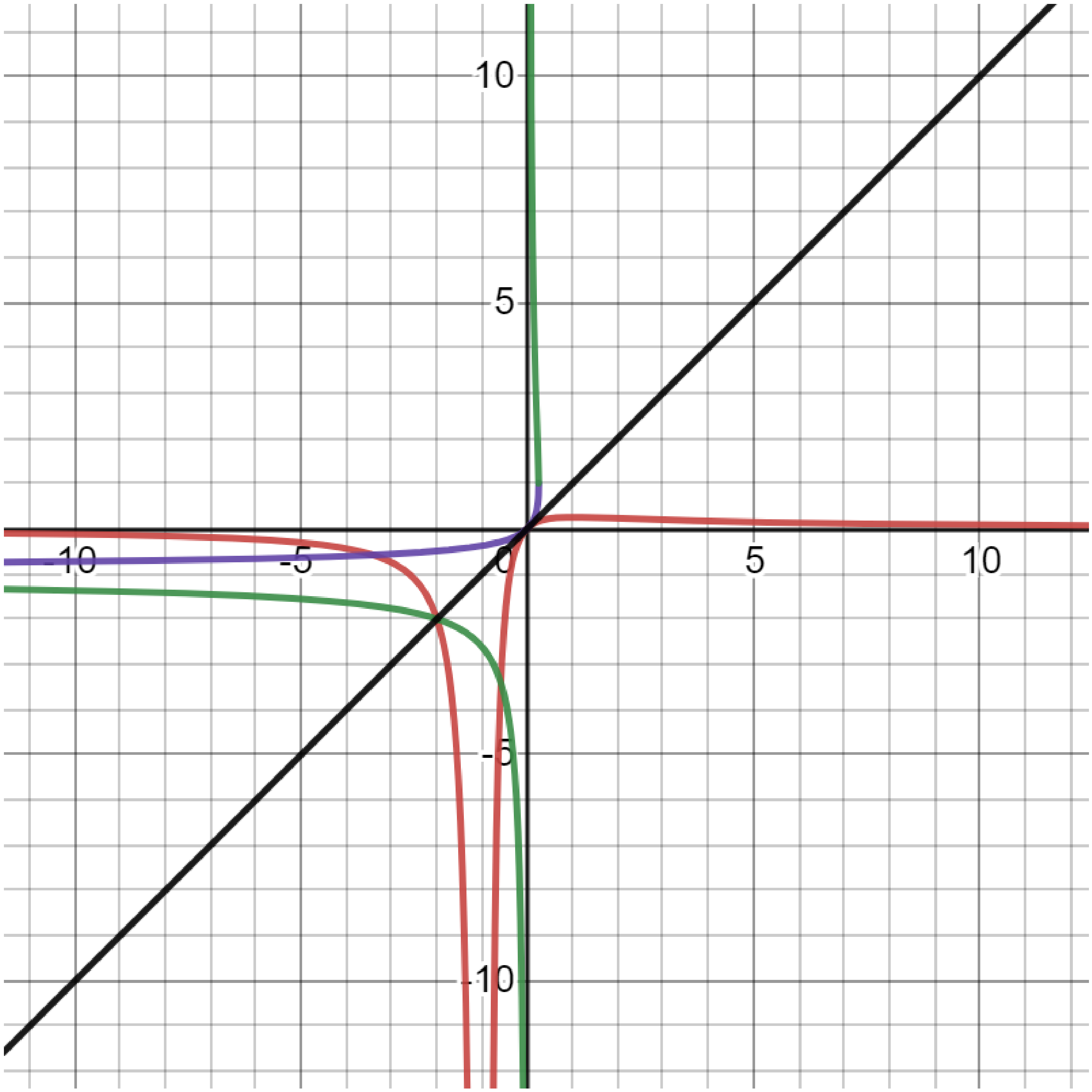
Above for b = c = 1.



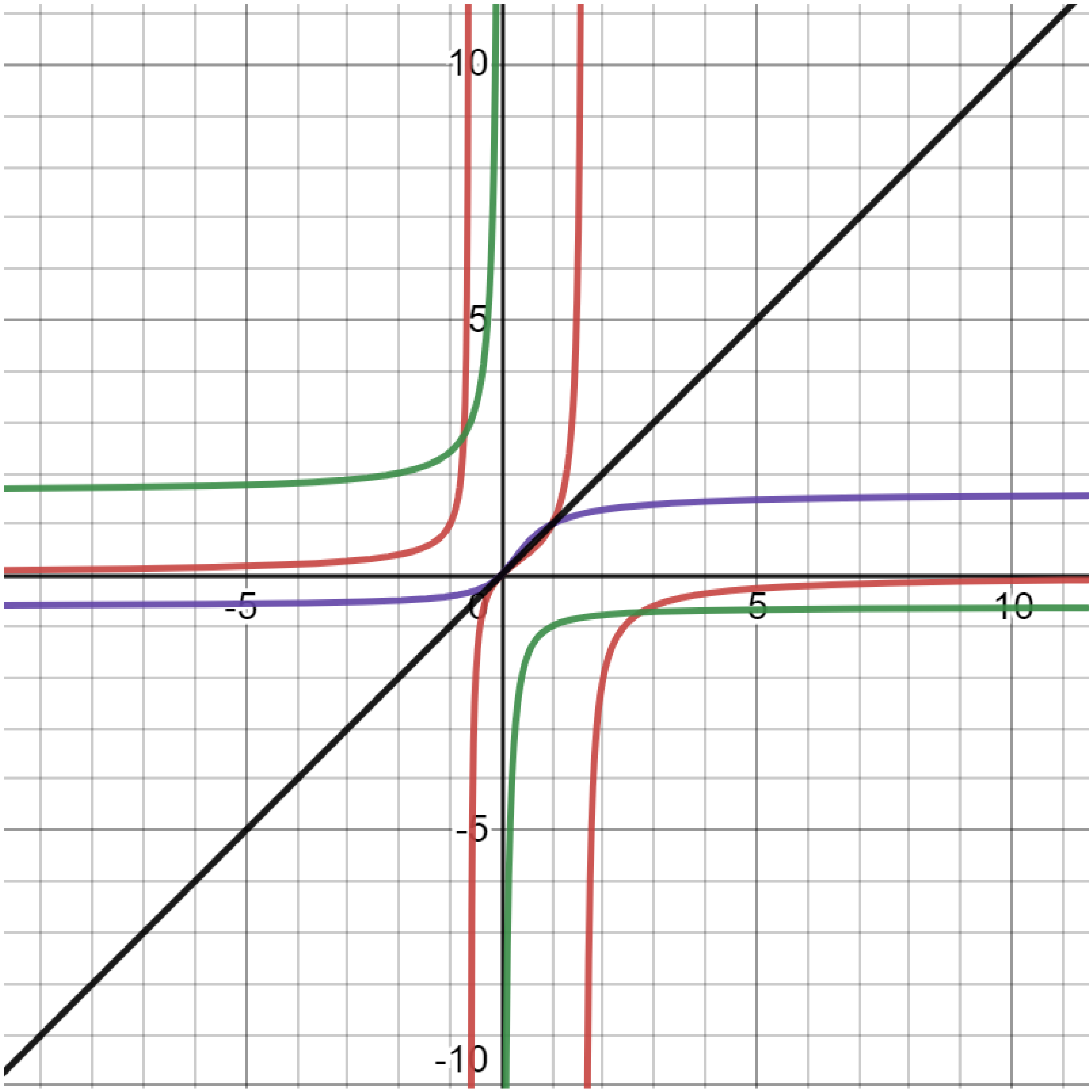
Above for b = -2.5 and c = 1.



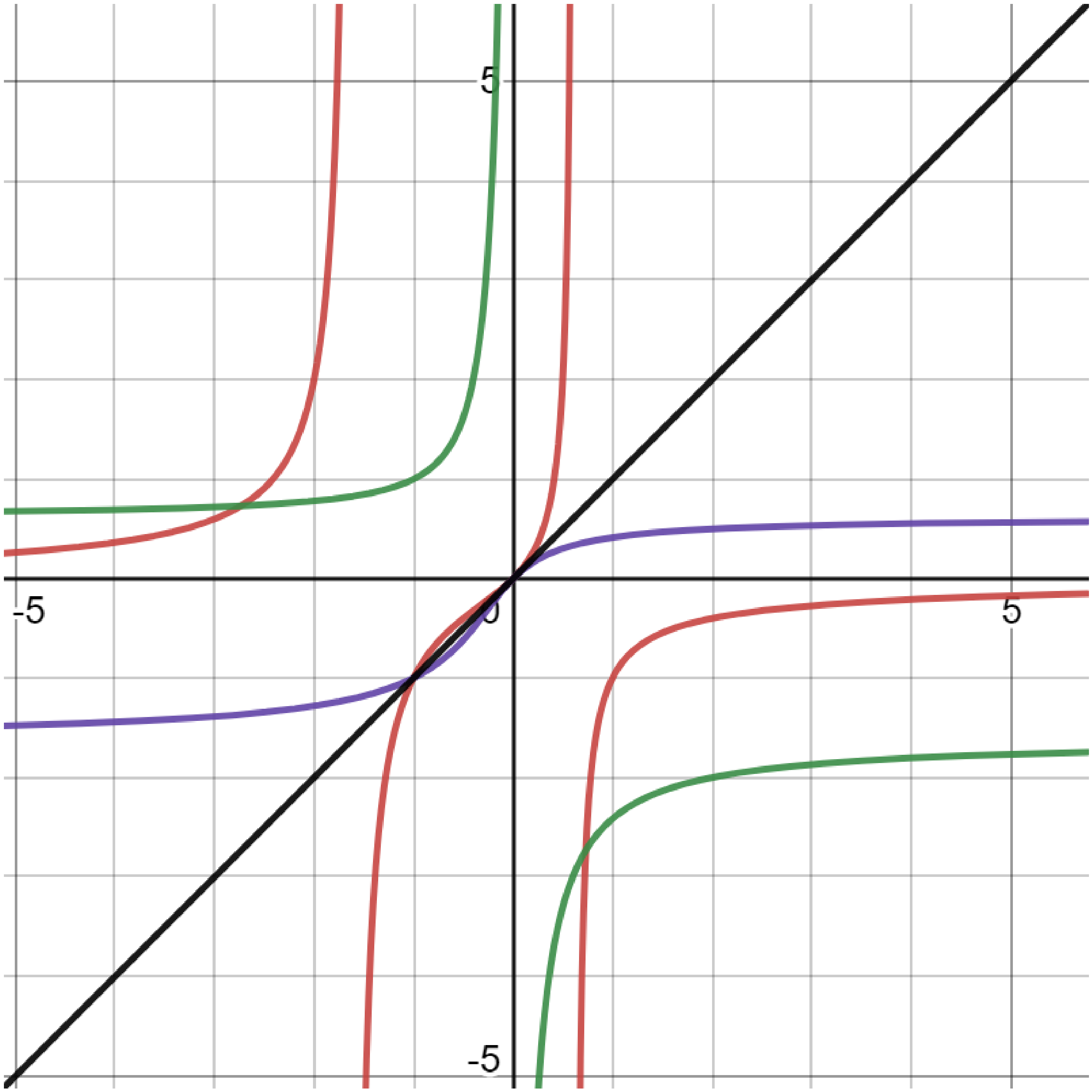
Above for b = -2.5 and c = 4.



Above for b = 2 and c = 1.



Above for b = 1 and c = -1.



Above for b = -1 and c = -1.

See also my post "A Taste of Moonshine in Free Moments" for a discussion of a particular case.