Composition, Conjugation, and the Umbral Calculus: Conjugates of the Bernoulli and hyperbinomial polynomials

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This set of notes consists of a review of the application of umbral conjugation to obtain relations among four core Sheffer polynomial sequences (SPSs)--the two binomial Stirling polynomials of the first and second kinds, $St1_n(x)$ and $St2_n(x)$, respectively, which are an inverse pair under umbral composition, and a pair of Appell SPSs, the Bernoulli polynomials $Ber_n(x)$ and their conjugate sequence $\overline{Ber}_n(x)$.

An additional pair of conjugate Appell sequences--the hyperbolic polynomials and the Pascal polynomials- associated with the Lambert W-function and its inverse te^t about the origin are also introduced.

To avoid an alphabetic morass, throughout these notes I use the convenient umbral notation and maneuver $(U.)^n = U_n$ to simplify arguments and clarify formulas. For more on umbral notation, see the post Umbral Mojo. I also use the same letter, lower and uppercase, to denote related quantities; e.g., $(A.(x))^n = A_n(x)$ refers to an Appell polynomial or the Appell polynomial sequence; $A(t) = e^{a.t} = \sum_{n\geq 0} \; a_n \frac{t^n}{n!}$, the exponential generating function (e.g.f) of the moments, $(a.)^n = a_n = A_n(0)$, of the Appell sequence; and $[A]$, the lower triangular infinite matrix of the coefficients of the Appell polynomials, columns progressing from lower order of x^n to higher, with elements $A_{n,k}$. The notation makes it easy to keep track of the five different but related constructs.

Any family of Sheffer polynomials (SPs)

$$(S.(x))^n = S_n(s) = \sum_{k=0}^{n} \; S_{n,k} \; x^k$$

has the e.g.f.

$$A(t)e^{B(t)x} = e^{S.(x)t},$$

formed from functions $A(t)$ and $B(t)$ analytic about the origin with $A(0) = 1$, $B(0) = 0$, and $D_{t=0} \; B(t) = dB(t)/dt \;|_{t=0} = B'(0) \neq 0$, and can be represented as an infinite lower triangular matrix $[S]$ with the elements $S_{n,k}$.

The Appell SPs, defined by

$$A_n(x) = (a. + x)^n = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}\, x^k,$$

are a normal subgroup of the SPs with $B(t) = t = B^{(-1}(t)$ and $A(t) = e^{a.t}$. The elements $A_{n,k}$ of their matrix rep $[A]$ are given by the lower triangular Pascal matrix diagonally multiplied by the moments $a_{m=n-k}$, i.e., $A_{n,k} = \binom{n}{k}\, a_{n-k}$. The first column is composed of the moments $a_m = A_m(0) = A_{m,0}$, and the elements of the main diagonal $A_{m,m}$ are all ones. The associated e.g.f. is

$$e^{A.(x)t} = e^{(a.+x)t} = e^{a.t}\, e^{xt} = A(t)\, e^{xt}.$$

Appell matrices can be expressed in terms of a matrix infinitesimal generator for the lower triangular Pascal matrix $[P]$ (OEIS A007318), which has the elements $P_{n,k} = \binom{n}{k}$ with the convention $\binom{n}{k} = 0$ for $k > n$. The infinitesimal generator is $[IgP] = \ln([P])$ (OEIS A132440), with $IgP_{n,k} = (n+1)\,\delta_{n-k-1}$, so the matrix rep of any Appell sequence can be expressed as

$$[A] = e^{a.[IgP]} = A([IgP]).$$

The infinitesimal generator $[IgP]$ is easily visualized as a lower triangular matrix with only one nonzero diagonal--the first subdiagonal $diag(1, 2, 3, 4, ...)$ composed of the natural numbers.

Consequently, any Appell matrix commutes with the infinitesimal generator $[IgP]$ of the Pascal matrix $[P] = e^{[IgP]}$ and with any other Appell matrix as well. This is reflected in the commutativity of umbral composition of any two families $A1_n(x)$ and $A2_n(x)$ of Appell polynomials.

In umbral notation,

$$A1_n(A2.(x)) = (a1. + A2.(x))^n = (a1. + (a2. + x)))^n = (a1. + a2. + x)^n$$

$$= (a2. + a1. + x)^n + (a2. + (a1. + x))^n = A2_n(A1.(x)).$$

This umbral witchcraft guides the tedious, nonintuitive, more conventional method of demonstrating the commutativity with sigma-binomial gymnastics and dummy variable transformations:

$S_n(x) = (a1. + a2. + x)^n = ((a1. + a2.) + x)^n = \sum_{k=0}^{n} \binom{n}{k} (a1. + a2.)^k \; x^{n-k}$

$= \sum_{k=0}^{n} x^{n-k} \binom{n}{k} \; \sum_{j=0}^{k} \binom{k}{j} \; a1_{k-j} \; a2_j$

$= n! \sum_{k=0}^{n} \frac{x^{n-k}}{(n-k)!} \; \sum_{j=0}^{k} \; \frac{a1_{k-j}}{(k-j)!} \; \frac{a2_j}{j!}$

$= n! \sum_{m=0}^{n} \frac{x^m}{m!} \; \sum_{j=0}^{n-m} \; \frac{a1_{n-m-j}}{(n-m-j)!} \; \frac{a2_j}{j!}$,

so, for $p > n$, $S_{n,p} = 0$, and, for $0 \leq p \leq n$,

$S_{n,p} = \sum_{j=0}^{n-p} \; \frac{a1_{n-p-j}}{(n-p-j)!} \; \frac{a2_j}{j!} = \sum_{j=0}^{n-p} \; \frac{a2_{n-p-j}}{(n-p-j)!} \; \frac{a1_j}{j!}.$

Now similarly perform the umbral composition via sigma summation:

$$A12_n(x) = A1_n(A2.(x)) = \sum_{m=0}^{n} \; A1_{n,m} \; A2_m(x)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a1_{n-m} \; A2_m(x)$$

$= \sum_{m=0}^{n} \; \binom{n}{m} \; a1_{n-m} \; \sum_{j=0}^{m} \binom{m}{j} \; a2_{m-j} \; x^j$

$= n! \sum_{m=0}^{n} \; \sum_{j=0}^{m} \; \frac{a1_{n-m}}{(n-m)!} \; \frac{a2_{m-j}}{(m-j)!} \; \frac{x^j}{j!}$

$= n! \sum_{j \geq 0} \; \sum_{m \geq 0} \; \frac{a1_{n-m}}{(n-m)!} \; \frac{a2_{m-j}}{(m-j)!} \; \frac{x^j}{j!},$

so, for $0 \leq p \leq n$,

$A12_{n,p} = n! \sum_{m=p}^{n} \; \frac{a1_{n-m}}{(n-m)!} \; \frac{a2_{m-p}}{(m-p)!}$

$= n! \sum_{j=0}^{n-p} \; \frac{a1_j}{j!} \; \frac{a2_{n-p-j}}{(n-p-j)!}$

$= n! \sum_{j=0}^{n-p} \; \frac{a2_j}{j!} \; \frac{a1_{n-p-j}}{(n-p-j)!} = A21_{n,p}.$

This conventional method inefficiently demonstrates the commutativity of the umbral composition of any pair of Appell sequences:

$$A1_n(A2.(x)) = A12_n(x) = A21_n(x) = A2_n(A1.(x)).$$

In terms of the associated Appell e.g.f.s,

$$e^{A1.(A2.(x))t} = e^{(a1.+A2.(x))t} = e^{(a1.+a2.+x)t}$$

$$= e^{(a1.+a2.)t}\ e^{xt} = e^{a1.t}\ e^{a2.t}\ e^{xt}$$

$$= A1(t)\ A2(t)e^{xt} = A2(t)\ A1(t)e^{xt} = e^{A2.(A1.(x))t}$$

$$= A12(t)\ e^{xt} = A21(t)\ e^{xt}.$$

Reprising, the commutativity of any pair of Appell umbral variables under multiplication and addition and the nature of Appell polynomials as binomial convolutions of their moments with the power monomials, i.e, $A_n(x) = (a.+x)^n$, imply the commutativity of the associated Appell coefficient matrix with the matrix infinitesimal generator of the Pascal matrix, the commutativity of the multiplication of the coefficient matrices of any pair of Appell sequences, and the commutativity of umbral composition of any two Appell sequences. Further, the polynomials resulting from umbral composition of two sequences of Appell polynomials are also an Appell sequence with moment e.g.f. given by the commutative product of the moment e.g.f.s of the initial two Appell sequences; i.e., $A12(t) = A21(t) = A1(t)A2(t)$.

Three of the most famous and ubiquitous Appell sequences are the prototypical Pascal polynomials $P_n(x) = (1+x)^n$, the Bernoulli polynomials $Ber_n(x) = (ber.+x)^n$, and the family of Hermite polynomials $Her_n(x) = (her.+x)$ with her_n being the number of perfect matchings for the vertices of the hypertetrahedra/hypertriangles, or n-simplices.

The binomial SPs, $B_n(x)$, are those SPs for which $A(t) = 1$; i.e., the associated e.g.f. Is

$$e^{B.(x)t} = e^{B(t)x}.$$

Their matrix rep has a first column of all zeros except for the first element $B_{0,0} = B_0(0) = 1$, i.e.,

$$B_{n,0} = B_n(0) = \delta_n,$$

and a main diagonal with elements

$$B_{m,m} = \frac{D^m_{x=0}}{m!}\frac{D^m_{t=0}}{m!}e^{B(t)x} = \frac{D^m_{t=0}}{m!}\ (B(t))^m.$$

The second column has the elements

$$B_{m,1} = D_{x=0}B_m(x) = B'_m(0) = D_{x=0}D^m_{t=0}e^{B.(x)t}$$

$$= D_{x=0}D^m_{t=0}e^{B(t)x} = D^m_{t=0}B(t) = B^{(m)}(0),$$

so contains the coefficients for formal Taylor series of $B(t)$ and the unique information to reconstruct the full binomial matrix or leading principal submatrices, analogous to the first column of an Appell matrix containing the formal Taylor series coefficients of $A(t)$ and, therefore, the unique information for constructing the full Appell matrix or leading principal submatrices.

A core maneuver in umbral calculus is the, in general noncommutative, umbral composition of two families of SPs, $S1_n(x)$ and $S2_n(x)$, which in the matrix rep is simply multiplication of the corresponding matrices. In umbral maneuvers, this corresponds to

$$S1_n(S2.(x)) = \sum_{k=0}^{n}\ S1_{n,k}\ (S2.(x))^k = \sum_{k=0}^{n}\ S1_{n,k}\ S2_k(x)$$

$$= \sum_{k=0}^{n}\ S1_{n,k}\ \sum_{j=0}^{k} S2_{k,j}x^j = S12_n(x).$$

Then the matrix $[S12]$ representing the umbral composition of the two SPs has the elements

$$S12_{n,m} = \sum_{k=0}^{n}\ S1_{n,k}\ S2_{k,m},$$

so the matrix equivalent of umbral composition is the matrix multiplication

$$[S12] = [S1][S2].$$

The Sheffer umbral compositional inverse sequence $\hat{S}_n(x)$ of a Sheffer sequence is defined by the multiplicative inverse of its matrix rep, i.e.

$$[\hat{S}] \cdot [S] = 1 = [S] \cdot [\hat{S}],$$

or, equivalently, by the umbral composition

$$S_n(\hat{S}.(x)) = x^n = \hat{S}_n(S.(x)).$$

In terms of e.g.f.s, with an Appell Sheffer sequence

$$e^{A.(x)t} = A(t)\; e^{xt},$$

then

$$e^{\hat{A}.(x)t} = \hat{A}(t)\; e^{xt} = \frac{1}{A(t)}\; e^{xt},$$

where $\hat{A}(t) = 1/A(t) = (A(t))^{-1}$, the multiplicative inverse of $A(t)$,

and with a binomial Sheffer sequence

$$e^{B.(x)t} = e^{B(t)x},$$

then

$$e^{\hat{B}.(x)t} = e^{\hat{B}(t)x} = e^{B^{(-1)}(t)x},$$

where $B^{(-1)}(t)$ is the compositional inverse of $B(t)$.

With umbral maneuvers, these claims become somewhat transparent. For an Appell sequence $A_n(x)$ and its umbral inverse sequence $\hat{A}_n(x)$,

$$e^{A.(\hat{A}.(x))t} = e^{(a.+\hat{A}.(x))t} = e^{a.t}e^{\hat{A}.(x)t} = A(t)\hat{A}(t)e^{xt},$$

and this reduces to e^{xt} so that $A_n(\hat{A}.(x)) = x^n$ when

$$\hat{A}(t) = 1/A(t).$$

For a binomial Sheffer sequence $B_n(x)$ and its umbral inverse sequence $\hat{B}_n(x)$,

$$e^{B.(\hat{B}.(x))t} = e^{\hat{B}.(x)B(t)} = e^{x\hat{B}(B(t))},$$

and this reduces to e^{xt} so that $B_n(\hat{B}.(x)) = x^n$ when

$\hat{B}(t) = B^{(-1)}(t)$.

The multiplicative or compositional inversion of two general classes of functions $A(t)$ and $B(t)$ that are analytic about the origin are intimately linked to matrix inversion, or, equivalently, umbral compositional inversion, by the above arguments .

This is easily generalized to finding the formal inverses of general formal power series not necessarily convergent

$A(t) = 1 + a_1 t + a_2 \frac{t^2}{2!} + ...$

and

$B(t) = b_1 t + b_2 \frac{t^2}{2!} + ...$.

Lying at the core of finding the formal multiplicative inverse of a formal Taylor series are the refined Euler characteristic partition polynomials, or refined face partition polynomial (RFPs) of the permuto/permutahedra (cf. OEIS A133314), and for the formal compositional inverse, the RFPs of the associahedra (cf. a normalization of A133437).

Umbral composition of an Appell Sheffer sequence with a binomial Sheffer sequence has the three reps

1) polynomial

$S_n(x) = AB_n(x) = A_n(B.(x))$

2) e.g.f.

$e^{S.(x)t} = e^{AB.(x)t} = e^{A.(B.(x))t} = e^{(a.+B.(x))t} = e^{a.t}e^{B.(x)t} = A(t)e^{xB(t)}$

3) matrix

$[S] = [AB] = [A][B]$.

Any Sheffer polynomial sequence has the e.g.f. given above.

Successive composition of a binomial Sheffer sequence with an Appell sequence composed with the umbral inverse of the binomial Sheffer sequence are conjugations with the reps

1) polynomial:

$$S_n(x) = \hat{B}AB_n(x) = \hat{B}_n(A.(B.(x)))$$

2) e.g.f.:

$$e^{S.(x)t} = e^{\hat{B}.(A.(B.(x)))t} = e^{A.(B.(x))\hat{B}(t)} = e^{(a.+B.(x))\hat{B}(t)}$$

$$= e^{a.\hat{B}(t)}e^{B.(x)\hat{B}(t)} = e^{a.\hat{B}(t)}e^{xB(\hat{B}(t))} = A(\hat{B}(t))e^{xt}$$

3) matrix:

$$[S] = [\hat{B}AB] = [\hat{B}][A][B]$$

$$= [B^{(-1)}AB] = [B^{(-1)}][A][B] = [B]^{-1}[A][B]$$

$$= [\hat{B}A\hat{B}^{(-1)}] = [\hat{B}][A][\hat{B}^{(-1)}] = [\hat{B}][A][\hat{B}]^{-1}.$$

Consequently, given $B(t)$ and its compositional inverse $\hat{B}(t) = B^{(-1)}(t)$, the two Appell sequences $A_n(x)$ and $\bar{A}_n(x)$ with e.g.f.s

$$e^{A.(x)t} = A(t)e^{xt} = \frac{t}{B(t)}e^{xt}$$

and

$$e^{\bar{A}.(x)t} = \bar{A}(t)e^{xt} = \frac{\hat{B}(t)}{t}e^{xt}$$

are related by the functional composition

$$\bar{A}(t) = \frac{\hat{B}(t)}{t} = A(\hat{B}(t))$$

and, therefore, as the umbral conjugation

$$\bar{A}_n(x) = \hat{B}_n(A.(B.(x))),$$

or, as the matrix conjugation

$$[\bar{A}] = [\hat{B}][A][B] = [B]^{-1}[A][B] = [\hat{B}][A][\hat{B}]^{-1},$$

so $A_n(x)$ and $\bar{A}_n(x)$ are a conjugate pair of Appell polynomial sequences.

The binomial Sheffer polynomials all vanish at $x = 0$ except for $B_0(x) = 1$; i.e., $B_n(0) = \delta_n$, so

$$\bar{a}_n = \bar{A}_n(0) = \hat{B}_n(A.(B.(0))) = \sum_{k=0}^{n} \hat{B}_{n,k}\ A_{k,0} = \sum_{k=0}^{n} \hat{B}_{n,k}\ a_k,$$

or, in terms of column vectors,

$[\bar{a}] = [\hat{B}][a]$,

and, conversely, since $[\hat{B}]^{-1} = [B]$,

$[a] = [B][\bar{a}]$.

As noted above, every Appell sequence $A_n(x)$ with the moment e.g.f. $A(t)$ also has an umbral inverse sequence, $\hat{A}_n(x)$ with the moment e.g.f. $\hat{A}(t) = 1/A(t)$, so, with the assignments above,

$$e^{A.(x)t} = A(t)\ e^{xt} = \tfrac{t}{B(t)}\ e^{xt}$$

with conjugate $\bar{A}_n(x)$ defined by

$$e^{\bar{A}.(x)t} = \bar{A}(t)\ e^{xt} = \tfrac{\hat{B}(t)}{t}\ e^{xt},$$

and umbral inverse sequence $\hat{A}_n(x)$, by

$$e^{\hat{A}.(x)t} = \hat{A}(t)\ e^{xt} = \tfrac{B(t)}{t}\ e^{xt}$$

with the Appell sequence $\bar{\hat{A}}_n(x)$ conjugate to $\hat{A}_n(x)$ defined by the e.g.f.

$$e^{\bar{\hat{A}}.(x)t} = \bar{\hat{A}}(t)\ e^{xt} = \frac{t}{\hat{B}(t)}\ e^{xt} = \hat{\bar{A}}(t)\ e^{xt} = e^{\hat{\bar{A}}.(x)t}.$$

Consequently, the umbral inverse of the conjugate is the conjugate of the umbral inverse, i.e.,

$$\hat{\bar{A}}_n(x) = \bar{\hat{A}}_n(x),$$

and

$$\hat{\bar{A}}(t) = \bar{\hat{A}}(t) = \frac{t}{\hat{B}(t)} = \hat{A}(\hat{B}(t)),$$

and, therefore, in parallel with the conjugations above,

$$\hat{\bar{A}}_n(x) = \bar{\hat{A}}_n(x) = \hat{B}_n(\hat{A}.(B.(x))),$$

or, in the matrix rep,

$$[\bar{\hat{A}}] = [\hat{B}][\hat{A}][B]$$

$$= [B]^{-1}[\hat{A}][B] = [\hat{B}][\hat{A}][\hat{B}]^{-1}$$

$$= [B]^{-1}[A]^{-1}[B] = [\hat{B}][A]^{-1}[\hat{B}]^{-1}$$

$$= ([A][B])^{-1}[B] = [\hat{B}]([\hat{B}][A])^{-1}$$

$$= ([B]^{-1}[A][B])^{-1} = ([\hat{B}][A][B])^{-1} = [\bar{A}]^{-1} = [\hat{\bar{A}}].$$

Check:

$$\bar{A}_n(\hat{\bar{A}}.(x)) = x^n$$

Is corroborated by

$$I = [\bar{A}][\bar{A}]^{-1} = [\bar{A}][\hat{\bar{A}}]$$

$$= [B]^{-1}[A][B] \cdot [B]^{-1}[\hat{A}][B] = [B]^{-1}[A][\hat{A}][B]$$

$$= [B]^{-1}[A][A]^{-1}[B] = I.$$

The Bernoulli numbers and polynomials via conjugation

For the Appell Bernoulli polynomial $Ber_n(x)$ and associated binomial Sheffer sequence, the Stirling polynomials of the second kind $St2_n(x)$,

$$Ber(t) = \frac{t}{e^t-1} = \frac{t}{St2(t)} = \frac{t}{B(t)}$$

and

$$St2(t) = e^t - 1 = B(t),$$

the umbral inverse sequences are defined with the multiplicative inverse

$$Rcp(t) = \widehat{Ber}(t) = \frac{1}{Ber(t)} = \frac{e^t-1}{t} = \frac{St2(t)}{t} = \frac{B(t)}{t}$$

and the compositional inverse

$$\widehat{St2}(t) = St2^{(-1)}(t) = \ln(1+t) = St1(t) = \hat{B}(t) = B^{(-1)}(t).$$

In terms of e.g.f.s, the umbral inverse pair of Appell sequences are defined by

$$e^{Ber.(x)t} = Ber(t)\ e^{xt} = \frac{t}{e^t-1}\ e^{xt}$$

and

$$e^{\widehat{Ber}.(x)t} = \widehat{Ber}(t)\ e^{xt} = \frac{1}{Ber(t)}\ e^{xt} = \frac{e^t-1}{t}\ e^{xt}$$

$$= e^{Rcp.(x)t} = \sum_{n\geq 0}\ \frac{(x+1)^{n+1}-x^{n+1}}{n+1}\ \frac{t^n}{n!}$$

and the umbral inverse pair of associated binomial sequences, by

$$e^{St2.(x)t} = e^{St2(t)x} = e^{(e^t-1)x} = e^{B(t)x}$$

and

$$e^{\widehat{St2}.(x)t} = e^{\widehat{St2}(t)x} = e^{\ln(1+t)x} = e^{St1(t)x} = e^{St1.(x)t} = e^{\hat{B}.(x)t} = e^{\hat{B}(t)x}$$

$$= (1+t)^x = \sum_{n\geq 0} \binom{x}{n} \; t^n = \sum_{n\geq 0} (x)_n \; \frac{t^n}{n!}.$$

The associated Appell conjugates w.r.t. $B(t) = St2(t)$ are given by

$$e^{\overline{Ber}.(x)t} = \frac{\ln(1+t)}{t} \; e^{xt} = \frac{St1(t)}{t} \; e^{xt}$$

and

$$e^{\overline{\widehat{Ber}}.(x)t} = e^{\overline{Rcp}.(x)t} = \frac{t}{\ln(1+t)} \; e^{xt} = \frac{t}{St1(t)} \; e^{xt}$$

with

$$\overline{Ber}_n(x) = \widehat{St2}_n(Ber.(St2.(x))) = St1_n(Ber.(St2.(x)))$$

and

$$\overline{\widehat{Ber}}_n(x) = \overline{Rcp}_n(x)$$

$$= \widehat{St2}_n(Rcp.(St2.(x))) = St1_n(Rcp.(St2.(x)))$$

$$= \overline{\widehat{Ber}}.(x) = St1_n(\widehat{Ber.(St2.(x))}) = \widehat{St2}_n(\widehat{Ber}.(\widehat{St1}.(x))),$$

or

$$[\overline{Ber}] = [\widehat{St2}][Ber][St2] = [St1][Ber][St2]$$

$$= [St2]^{-1}[Ber][St2] = [St1][Ber][St1]^{-1}$$

and

$$[\overline{\widehat{Ber}}] = [\overline{Rcp}]$$

$$= [\widehat{St2}][Rcp][St2] = [St1][Rcp][St2]$$

$$= [\widehat{\overline{Ber}}] = [\overline{Ber}]^{-1} = ([St1][Ber][St2])^{-1} = [St2]^{-1}[Ber]^{-1}[St1]^{-1}.$$

Naturally, several converse relations also hold such as

$$[Rcp] = [St2][\overline{Rcp}][St1] = [\widehat{St1}][\overline{Rcp}][St1] = [St2][\overline{Rcp}][\widehat{St2}]$$

$$= [St1]^{-1}[\overline{Rcp}][St2]^{-1} = [St1]^{-1}[\overline{Rcp}][St1] = [St2][\overline{Rcp}][St2]^{-1}.$$

From the general arguments above for the evaluation of the conjugations at $x = 0$,

$$\overline{ber}_n = (-1)^n \; \frac{n!}{n+1} = \sum_{k=0}^{n} \; St1_{n,k} \; ber_k,$$

or

$$[\overline{ber}] = [St1][ber].$$

Conversely,

$$[ber] = [St2][\overline{ber}],$$

or

$$b_n = \sum_{k=0}^{n} St2_{n,k} \; (-1)^k \frac{k!}{k+1} = \sum_{k=0}^{n} (-1)^k \frac{Perm_{n,k}}{k+1},$$

where

$$Perm_{n,k} = k! \; St2_{n,k}$$

enumerate the distinct faces of the permutahedra, cf. OEIS A019538, A049019, and A133314.

The first few rows of the relevant polynomials are

For $[St2]$, (cf. A048993 and the A008277, the Stirling polynomials of the second kind, the Bell-Touchard-Steffensen-exponential polynomials)

$e^{St2.(x)t} = e^{(e^t-1)x}$ and

$St2_0(x) = 1$

$St2_1(x) = x$

$St2_2(x) = x + x^2$

$St2_3(x) = x + 3x^2 + x^3$

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for $[St1] = [St2]^{-1} = [\widehat{St2}]$,

(cf. A048994 and the variant A008275, the Stirling polynomials of the first kind, the falling factorials, see also the Pocchammer symbol)

$e^{St1.(x)t} = e^{\ln(1+t)x} = (1+t)^x$ and

$St1_0(x) = 1$

$St1_1(x) = x$

$St1_2(x) = x(x-1) = -x + x^2$

$St1_3(x) = x(x-1)(x-2) = 2x - 3x^2 + x^3$

For $[Ber]$,

the Bernoulli polynomials,

$e^{Ber.(x)t} = Ber(t)\ e^{xt} = \frac{t}{e^t-1}\ e^{xt}$ and

$ber_0 = 1$, $Ber_0(x) = 1$

$ber_1 = -\frac{1}{2}$, $Ber_1(x) = \frac{-1+2x}{2} = -\frac{1}{2} + x$

$ber_2 = \frac{1}{6}$, $Ber_2(x) = \frac{1-6x+6x^2}{6} = \frac{1}{6} - x + x^2$

$ber_3 = 0$, $Ber_3(x) = \frac{x-3x^2+2x^3}{2} = 0 + \frac{1}{2}x - \frac{3}{2}x^2 + x^3$

For $[Rcp] = [Ber]^{-1} = [\widehat{Ber}]$,

the reciprocal polynomials,

$e^{Rcp.(x)t} = \frac{e^t-1}{t}\ e^{xt} = \sum_{n\geq 0}\ \frac{(x+1)^{n+1}-x^{n+1}}{n+1}\ \frac{t^n}{n!}$ and

$rcp_0 = 1$, $Rcp_0(x) = 1$

$rcp_1 = \frac{1}{2}$, $Rcp_1(x) = \frac{1+2x}{2} = \frac{1}{2} + x$

$rcp_2 = \frac{1}{3}$, $Rcp_2(x) = \frac{1+3x+3x^2}{3} = \frac{1}{3} + x + x^2$

$rcp_3 = \frac{1}{4}$, $Rcp_3(x) = \frac{1+4x+6x^2+4x^3}{4} = \frac{1}{4} + x + \frac{3}{2}x^2 + x^3$

For $[\overline{Ber}] = [\widehat{St2}][Ber][St2] = [St1][Ber][St2]$,

the conjugate Bernoulli polynomials,

$e^{\overline{Ber}.(x)t} = \frac{\ln(1+t)}{t}\ e^{xt} = \frac{St1(t)}{t}\ e^{xt}$ and

$\overline{ber}_0 = 1$, $\overline{Ber}_0(x) = 1$

$\overline{ber}_1 = -\frac{1}{2}$, $\overline{Ber}_1(x) = \frac{-1+2x}{2} = -\frac{1}{2} + x$

$\overline{ber}_2 = \frac{2}{3}$, $\overline{Ber}_2(x) = \frac{2-3x+3x^2}{3} = \frac{2}{3} - x + x^2$

$\overline{ber}_3 = -3/2$, $\overline{Ber}_3(x) = \frac{-3+4x-3x^2+2x^3}{2} = -\frac{3}{2} + 2x - \frac{3}{2}x^2 + x^3$

For $[\overline{Rcp}] = [St1][Rcp][St2] = [\overline{Ber}]^{-1} = [\widehat{\overline{Ber}}] = [\overline{\widehat{Ber}}]$,

the conjugate reciprocal polynomials,

$e^{\overline{Rcp.(x)t}} = \frac{t}{\ln(1+t)}\ e^{xt}$ and

$\overline{rcp}_0 = 1$, $\overline{Rcp}_0(x) = 1$

$\overline{rcp}_1 = \frac{1}{2}$, $\overline{Rcp}_1(x) = \frac{1+2x}{2} = \frac{1}{2} + x$

$\overline{rcp}_2 = -\frac{1}{6}$, $\overline{Rcp}_2(x) = \frac{-1+6x+6x^2}{6} = -\frac{1}{6} + x + x^2$

$\overline{rcp}_3 = \frac{1}{4}$, $\overline{Rcp}_3(x) = \frac{1-2x+6x^2+4x^3}{4} = \frac{1}{4} - \frac{1}{2}x + \frac{3}{2}x^2 + x^3$

Spot check:

$\overline{Ber}_2(x) = St1_2(Ber.(St2.(x))) = -Ber_1(St2.(x)) + Ber_2(St2.(x))$

$= -(\frac{-1+2St2_1(x)}{2}) + \frac{1-6St2_1(x)+6St2_2(x)}{6}$

$= -(\frac{-1+2x}{2}) + \frac{1-6x+6(x+x^2)}{6}$

$= \frac{2}{3} - x + x^2 = (\overline{ber}. + x)^2.$

Spot check:

$ber_2 = St2_2(\overline{ber}.) = \sum_{k=0}^{2} St2_{2,k}\ (-1)^k \frac{k!}{k+1}$

$= \overline{ber}_1 + \overline{ber}_2 = -\frac{1}{2} + \frac{2}{3} = \frac{1}{6}.$

Spot check:

$$\overline{Rcp}_2(x) = St1_2(Rcp.(St2.(x))) = -Rcp_1(St2.(x)) + Rcp_2(St2.(x))$$

$$= -(\tfrac{1}{2} + St2_1(x)) + \tfrac{1}{3} + St2_1(x) + St2_2(x)$$

$$= -\tfrac{1}{6} + St2_2(x) = -\tfrac{1}{6} + x + x^2.$$

The hyperbinomial and Pascal polynomials as conjugates

As a further illustration of the umbral conjugation algebra, consider $B(t) = te^t$ and its inverse, about the origin, $\hat{B}(t) = B^{(-1)}(t) = LW(t)$, the principle Lambert W-function (cf. Wikipedia, MathWorld).

The coefficients of the binomial Sheffer polynomials $Idp_n(x)$ associated with $B(t) = Idp(t) = te^t$ comprise the integral matrix for the idempotent numbers of A059297. The associated e.g.f. is

$$e^{B(t)x} = e^{Idt(t)x} = e^{te^t x} = e^{Idp.(x)t}.$$

The associated Appell sequence $A_n(x)$ then has the e.g.f.

$$e^{A.(x)t} = \frac{t}{B(t)} e^{xt} = \frac{t}{Idp(t)} e^{xt} = e^{-t} e^{xt} = e^{(x-1)t} = e^{Ps.(x)t}$$

with the associated moments $a_n = ps_n = (-1)^n$ and polynomials

$$A_n(x) = (a. + x)^n = (x + ps.)^n = (x-1)^n = Ps_n(x),$$

giving the signed Pascal matrix A007318 with

$$Ps_{n,k} = \binom{n}{k}\, ps_{n-k} = \binom{n}{k}\, (-1)^{n-k}.$$

The umbral inverse binomial Sheffer sequence to $B_n(x) = Idp_n(x)$ is the Abel polynomial sequence $\hat{B}_n(x) = Ab_n(x) = x(x-n)^{n-1}$ of A137452 with the e.g.f.

$$e^{\hat{B}(t)x} = e^{B^{(-1)}(t)x} = e^{Idt^{(-1)}(t)x} = e^{LW(t)x} = e^{Ab.(x)t}.$$

The Appell polynomials $\bar{A}_n(x) = (\bar{a}. + x)^n$ conjugate to $A_n(x)$ w.r.t. $B(t) = Idp(t) = te^t$ are the hyperbinomial polynomials $Hb_n(x) = (hb. + x)^n$ of signed A088956 with the e.g.f.

$$e^{\bar{A}.(x)t} = \frac{\hat{B}(t)}{t}e^{xt} = \frac{B^{(-1)}(t)}{t}e^{xt} = \frac{Idp^{(-1)}(t)}{t}e^{xt} = \frac{LW(t)}{t}e^{xt} = e^{Hb.(x)t} = Hb(t)e^{tx}$$

and the moments $[\bar{a}]^T = [hb]^T = [1,-1,3,-16,125,...]$, which is the sequence signed A000272 excepting the initial one, i.e., $hb_n = (-1)^n A00272(n+1)$.

The general matrix conjugation relation

$$[\bar{A}] = [\hat{B}][A][B]$$

implies

$$[\bar{a}] = [\hat{B}][a],$$

which, in this case, become

$$[Hb] = [Ab][Ps][Idp]$$

$$= [signed\ A088956] = [A137452][signed\ A007318][A059297]$$

and

$$[hb] = [Ab][ps] = [signed\ A000272\ excepting\ initial\ 1]$$

$$= [A137452][1,-1,1,-1,1,...]^T,$$

correctly implying the row sums of the unsigned matrix A137452 are A000272 excepting the first element.

The basic conjugation relation and the first few rows of the matrices for numerical checks are,

$[Hb] = [Ab][Ps][Idp]$

$= [signed\ A088956] = [A137452][signed\ A007318][A059297]$

$$\Rightarrow \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 3 & -2 & 1 & \\ -16 & 9 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & -2 & 1 & \\ 0 & 9 & -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 1 & -2 & 1 & \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 3 & 6 & 1 \end{bmatrix}.$$

The umbral inverse Appell Sheffer sequence to $A_n(x) = Ps_n(x)$ is $\hat{A}_n(x) = (-1)^n Ps_n(-x) = P_n(x) = (1+x)^n = \hat{P}s_n(x)$, the row polynomials of the Pascal matrix A007318, with the moments $\hat{a}_n(x) = 1$ and e.g.f.

$$e^{\hat{A}.(x)t} = \hat{A}(t)e^{xt} = \frac{1}{A(t)}e^{xt} = \frac{B(t)}{t}e^{xt} = e^t e^{xt} = e^{(x+1)t} = e^{P.(x)t}.$$

The associated Pascal matrix $[P] = [A007318]$ is the inverse of $[Ps]$, associated with $A_n(x) = Ps_n(x)$, i.e., $[P] = [Ps]^{-1}$.

The conjugate of $\hat{A}_n(x) = P_n(x)$ w.r.t. $B(t) = te^t$ is $\bar{\hat{A}}_n(x) = \hat{\hat{A}}_n(x) = RHb_n(x) = (rhb. + x)^n$, the reciprocal hyperbinomial polynomials with $[rhb]^T = [1, 1, -4, -27, ...] =$ A177885, or $rhb_n = (1-n)^{n-1}$ and with the e.g.f.

$$e^{\bar{\hat{A}}.(x)t} = e^{\hat{\hat{A}}.(x)t} = \frac{t}{\hat{B}(t)}e^{xt}$$

$$= \frac{t}{B^{(-1)}(t)}e^{xt} = \frac{t}{LW(t)}e^{xt} = e^{RHb.(x)t}.$$

The associated coefficient matrix $[RHb]$ is a differently signed version of A215534 and is given by the matrix conjugation

$[\hat{\hat{A}}] = [\bar{\hat{A}}] = [\hat{B}][\hat{A}][B]$

$= [RHb] = [Ab][P][Idp]$

$= [sign\ modified\ A215534]$

$= [A137452][A007318][A059297] = [A137452][A154372],$

implying

$[\hat{a}] = [\bar{\hat{a}}] = [\hat{B}][\hat{a}]$

$= [A177885] = [A137452][1, 1, 1, ...]^T,$

giving the row sums of A137452 correctly as A177885.

The above matrix conjugations imply

$[P]^{-n} = ([P]^{-1})^n = [Ps]^n = [Idp][Hb]^n[Ab]$

$= [A007318]^{-n} = [A059297][signed\ A088956]^n[A137452]].$

and

$[P]^n = [Idp][RHb]^n[Ab]$

$= [A007318]^n = [A059297][sign\ modified\ A215534]^n[A137452]].$

The e.g.f. associated with $[A007318]^m$ with m any integer is $e^{mt}e^{xt} = e^{(x+m)t}$.

Defining the natural logarithm of a lower triangular matrix with a diagonal of all ones as

$\ln(M) = \ln(I + (M - I)) = \sum_{n\geq 0}(-1)^n \frac{(M-I)^{n+1}}{n+1},$

then $\ln([P]) = [IGP] =$ [A132440], the infinitesimal generator of the Pascal matrix and

$\ln([P]^m) = m\ \ln([P]) = m\ [IGP].$

This allows extension of m from integers to complex numbers or even matrices that commute with P, as noted by Peter Bala; e.g., for real r, define the interpolated matrix

$[P]^r = \exp(r\ \ln([P]))$

consistent with the associated e.g.f.

$$(e^t)^r\ e^{xt} = e^{rt}e^{xt} = e^{(r+x)t}$$

and providing an interpretation of non-integer 'iterated' umbral composition. These extensions hold with $[P]$ replaced by a general Appell matrix $[A]$ and the e.g.f. e^t by the Appell moment generating function $A(t) = e^{a.t}$. In addition, r can be generalized to any Appell matrix since Appell matrices commute as shown below.

For easy comparison with the unsigned hyperbinomial matrix of A088956 and explorations of some relations in that entry noted by Peter Bala and in the reference by Gottfried Helms, consider

$$\widetilde{Hb}_n(x) = (x + \widetilde{hb})^n = (-1)^n\ Hb_n(-x) = (x - hb)^n$$

with the Appell e.g.f.

$$e^{\widetilde{Hb}.(x)t} = \widetilde{Hb}(t)\ e^{xt} = Hb(-t)\ e^{xt} = \frac{-LW(-t)}{t}\ e^{xt}$$

$$= \frac{Tr(t)}{t}\ e^{xt} = e^{-Tr(t)}\ e^{xt}$$

where $Tr(t))$ is the tree function presented in A000169 and the moments are $[\widetilde{hb}]^T = [1, 1, 3, 16, ...]$, or $\widetilde{hb}_n = A000272(n+1) = (n+1)^{n-1}$. This then is our new conjugate Appell sequence with e.g.f.

$$e^{\bar{A}.(x)} = \bar{A}(t)\ e^{xt} = e^{\bar{a}.t}\ e^{xt} = e^{(\bar{a}.+x)t}$$

$$= e^{\widetilde{Hb}.(x)} = \widetilde{Hb}(t)\ e^{xt} = e^{\widetilde{hb}.t}\ e^{xt} = e^{(\widetilde{hb}.+x)t}$$

$$= \frac{-LW(-t)}{t}\ e^{xt} = \frac{\hat{B}(t)}{t}\ e^{xt} = \frac{B^{(-1)}(t)}{t}\ e^{xt}$$

with respect to

$$B(t) = \hat{B}^{(-1)}(t) = (-LW(-t))^{(-1)} = te^{-t}.$$

Consequently, the new sequence $B_n(x)$ has the e.g.f.

$$e^{B.(x)t} = e^{te^{-t}x} = e^{-Idp.(-x)t}$$

with

$$B_n(x) = (-1)^n \; Idp_n(-x) = \widetilde{Idp}_n(x).$$

The new sequence $\hat{B}_n(x)$ has the e.g.f.

$$e^{\hat{B}.(x)t} = e^{-LW(-t)x} = e^{-Ab.(-x)t}$$

with

$$\hat{B}_n(x) = (-1)^n \; Ab_n(-x) = \widetilde{Ab}_n(x).$$

The new sequence $A_n(x)$ has the e.g.f.

$$e^{A.(x)} = \tfrac{t}{B(t)} e^{xt} = \tfrac{t}{te^{-t}} e^{xt} = e^t e^{xt} = e^{(1+x)t}$$

with

$$A_n(x) = (1+x)^n = P_n(x),$$

the row polynomials of the Pascal matrix.

The basic conjugation relation and the first few rows of the matrices for numerical checks are

$$[\widetilde{Hb}] = [\widetilde{Ab}][P][\widetilde{Idp}]$$

$$= [A088956] = [unsigned\ A137452][A007318][signed A059297]$$

$$\Rightarrow \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 3 & 2 & 1 & \\ 16 & 9 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 9 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & -2 & 1 & \\ 0 & 3 & -6 & 1 \end{bmatrix}.$$

From the basic conjugation relation, we have the matrix identity

$$[\widetilde{Hb}[\widetilde{Ab}] = [\widetilde{Ab}][P],$$

which, as an umbral composition, is the identity

$$\widetilde{Hb}_n(\widetilde{Ab}.(x)) = \widetilde{Ab}_n(P.(x)) = \widetilde{Ab}_n(1+x),$$

(this last identity is noted by Bala).

The basic conjugation relation also implies the relation between the two infinitesimal generators for the conjugate pair of Appell sequences

$$\ln([\widetilde{Hb}]) = [\widetilde{Ab}]\ln([P])[\widetilde{Idp}]$$

$$= [Ig\widetilde{Hb}] = [\widetilde{Ab}][IgP][\widetilde{Idp}].$$

As noted above, the infinitesimal generator for the Pascal matrix is presented in A132440, with nilpotent principal submatrices. The first few rows are

$$[IgP] = \ln([P]) \rightarrow$$

$$\begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 2 & 0 & \\ 0 & 0 & 3 & 0 \end{bmatrix}.$$

The matrix for any Appell sequence may expressed as

$$[A] = e^{a.[IgP]} = A([IgP]);$$

consequently, any Appell matrix commutes with $[IgP]$.

Bala states that the infinitesimal generator for the unsigned hyperbinomial transform is given by

$$[Ig\widetilde{Hb}] = [IgP][\widetilde{Hg}],$$

but doesn't give a proof.

The identity follows from a well-known identity of the tree function $Tr(t)$ presented in A000169 and related to the Lambert W-function by $Tr(t) = -LW(-x) = \hat{B}(t) = B^{(-1)}(t)$, so $Tr(t)$ is the compositional inverse of $B(t) = t\ e^{-t}$, implying, on substitution of $B^{(-1)}(t) = Tr(t)$ in this last set of equalities,

$$Tr(t) = t\ e^{Tr(t)}.$$

Consequently,

$$e^{Tr(t)} = \frac{Tr(t)}{t} = \frac{-LW(-1)}{t} = \frac{\hat{B}(t)}{t} = \widetilde{Hb}(t),$$

so

$$[\widetilde{Hb}] = \widetilde{Hb}([IgP]) = e^{Tr([Igp])}$$

and

$$[Ig\widetilde{Hb}] = \ln([\widetilde{Hb}]) = Tr([Igp]) = [IgP]\ e^{Tr([Igp])} = [IgP]\ [\widetilde{Hb}] = [\widetilde{Hb}]\ [IgP].$$

This is corroborated by noting that the product

$$[\widetilde{Hb}]\ [IgP]$$

corresponds to taking the derivative of the row polynomials $\widetilde{Hb}_n(x)$.

$$Tr(t) = t\ e^{Tr(t)} = e^{tr.t} = \sum_{n=0} tr_n\ \frac{t^n}{n!} = \sum_{n=0} n^{n-1}\frac{t^n}{n!}$$

with $tr_0 = 0$ and $tr_n = n^{n-1}$ otherwise. Then

$$[Tr]^T = [0, 1, 2, 9, 64, 625, ...] = [0, A000169].$$

This tree number sequence is related to (excepting the first element of the entry) $[A000272] \rightarrow [1, 1, 3, 16, 125, 1296, ...]$, the moments $\widetilde{Hb}_n(0) = hb_n = (n+1)^{n-1}$ of the hyperbinomial polynomials and the elements of the first column of the associated coefficient matrix $[\widetilde{Hb}] = e^{hb.[IgP]}$, via

$$\frac{d}{dx}\ \widetilde{Hb}_{n+1}(x)\ |_{x=0} = \frac{d}{dx}\ (\widetilde{hb}.+x)^{n+1}\ |_{x=0} = (n+1)\ (\widetilde{hb}.+x)^{n}|_{x=0}$$

$$= (n+1)\ \widetilde{hb}_n = (n+1)\ (n+1)^{n-1} = (n+1)^n = tr_{n+1},$$

for $n \geq 0, \text{ and } \frac{d}{dx}\ \widetilde{Hb}_0(x)\ |_{x=0} = 0 = tr_0$.

Reprising,

$$[Ig\widetilde{Hb}] = \ln([\widetilde{Hb}]) = Tr([IgP]) = [IgP]\ e^{Tr([IgP])}$$

$$= [IgP]\ [\widetilde{Hb}] = [\widetilde{Hb}]\ [IgP] = [\widetilde{Ab}][IgP][\widetilde{Idp}].$$

Calculating the first few rows of the infinitesimal generator from the last two matrix products or by taking the derivative of the row polynomials gives

$$[Ig\widetilde{Hb}] \rightarrow$$

$$\begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ 2 & 2 & 0 & \\ 9 & 6 & 3 & 0 \end{bmatrix}.$$

Then exponentiating and using $Tr(t) = t\ e^{Tr(t)}$ and $P(t) = e^t$ gives

$$[\widetilde{Hb}] = e^{[Ig\widetilde{Hb}]} = P([Ig\widetilde{Hb}])$$

$$= e^{Tr([IgP])} = \frac{Tr([IgP]}{[IgP]} = \widetilde{Hb}([IgP])$$

$$= \exp([IgP]\ e^{Tr([Igp])}) = \exp([IgP]\ [\widetilde{Hb}]) = \exp([\widetilde{Hb}]\ [IgP])$$

$$= \exp(\ln([P])\ [\widetilde{Hb}]) = \exp([\widetilde{Hb}]\ln([P])) = [P]^{[\widetilde{Hb}]}$$

$$= [\widetilde{Ab}]e^{[IgP]}[\widetilde{Idp}] = [\widetilde{Ab}][P][\widetilde{Idp}].$$

As noted by Bala and Helms, via repeated application of $[\widetilde{Hb}] = [P]^{[\widetilde{Hb}]}$, we obtain the power tower

$$[\widetilde{Hb}] = [P]^{[P]^{\cdots}}.$$

By conjugation, we obtain the dual sets of equalities

$$[IgP] = \ln([P]) = Tr([Ig\widetilde{Hb}]) = [Ig\widetilde{Hb}]\ e^{Tr([Ig\widetilde{Hb}])}$$

$$= [\widetilde{Idp}][Ig\widetilde{Hb}][\widetilde{Ab}].$$

and

$$[P] = e^{[IgP]} = P([IgP])$$

$$= e^{Tr([Ig\widetilde{Hb}])} = \frac{Tr([Ig\widetilde{Hb}]}{[Ig\widetilde{Hb}]} = \widetilde{Hb}([Ig\widetilde{Hb}])$$

$$= \exp([Ig\widetilde{Hb}]\ e^{Tr([Ig\widetilde{Hb}])}) = [\widetilde{Idp}]e^{[Ig\widetilde{Hb}]}[\widetilde{Ab}] = [\widetilde{Idp}][\widetilde{Hb}][\widetilde{Ab}].$$

We have $\ln([\widetilde{Hb}]) = [IgP][\widetilde{Hb}]$, so

$$\ln([\widetilde{Hb}])[\widehat{\widetilde{Hb}}] = \ln([\widetilde{Hb}])[\widetilde{Hb}]^{-1} = [IgP]$$

and exponentiation gives

$$[P] = \exp(\ln([\widetilde{Hb}])[\widehat{\widetilde{Hb}}]) = [\widetilde{Hb}]^{[\widehat{\widetilde{Hb}}]} = [\widetilde{Hb}]^{[\widetilde{Hb}]^{-1}}.$$

From the general arguments above, the e.g.f. for the Appell sequence $\widehat{\widetilde{Hb}}_n(x)$ with the associated matrix $[\widehat{\widetilde{Hb}}_n(x)] =$ [A215534], which is the umbral inverse of $\widetilde{Hb}_n(x)$, is

$$e^{\widehat{\widetilde{Hb}}.(x)t} = \frac{1}{\widetilde{Hb}(t)}\ e^{xt} = \frac{t}{Tr(t)}\ e^{xt} = e^{-Tr(t)}\ e^{xt}.$$

Additional identities among the Sheffer polynomials

Some additional identities among sets of conjugate and umbral inverse pairs of Sheffer polynomials such as those above were given in the notes “Reciprocal Sheffer Sextuplets and Conjugation Isomorphism” in my post on “Reciprocity and Umbral Witchcraft: An Eve with Stirling, Bernoulli, Archimedes, Euler, Laguerre, and Worpitzky” on my WordPress math blog Shadows of Simplicity. These additional identities are obtained by taking derivatives of the e.g.f.s of the binomial SPs;

$$\frac{D_x}{t}\ e^{B(t)x} = \frac{B(t)}{t}\ e^{B(t)x} = \hat{A}(t)\ e^{B(t)x}$$

$$= e^{\hat{a}.t}\ e^{B.(x)t} = e^{(\hat{a}.+B.(x))t} = e^{\hat{A}.(B.(x))t}$$

and

$$\frac{D_x}{t}\ e^{B(t)x}\ |_{x=0} = \frac{B(t)}{t} = \hat{A}(t) = e^{\hat{a}.t} = e^{\hat{A}.(B.(0))t}.$$

In addition,

$$\frac{B(t)}{t}\ e^{B(t)x} = \frac{t}{\hat{B}(t)}\ e^{xt}\ |_{t=B(t)} = \bar{\hat{A}}(t)\ e^{xt}\ |_{t=B(t)}$$

$$= \bar{\hat{A}}(B(t))\ e^{xB(t)}$$

and

$$\frac{D_x}{t}\ e^{B(t)x} = \sum_{n\geq 0} \frac{B'_{n+1}(x)}{n+1}\ \frac{t^n}{n!},$$

so

$$\frac{B'_{n+1}(x)}{n+1} = \hat{A}_n(B.(x)) = \hat{A}B_n(x)$$

with the matrix rep

$$[\hat{A}B] = [\hat{A}][B],$$

for which the first column has the e.g.f.

$$\bar{\hat{A}}(B(t)) = \frac{B(t)}{t} = \hat{A}(t) = e^{\hat{a}.t} = e^{\hat{A}.(B.(0))t}$$

with components

$$[\hat{A}B]_{n,0} = \frac{B'_{n+1}(0)}{n+1} = \frac{B_{n+1,1}}{n+1} = \hat{a}_n = \hat{A}_n(0) = \hat{A}_n(B.(0)).$$

This along with the conjugation formula

$$[\bar{\hat{A}}] = [\hat{B}][\hat{A}][B]$$

imply

$$\bar{\hat{A}}_{n,0} = \bar{\hat{a}}_n = \sum_{k=0}^{n} \hat{B}_{n,k}\ \hat{a}_k = \sum_{k=0}^{n} \hat{B}_{n,k}\ \frac{B_{n+1,1}}{n+1}.$$

Conversely,

$$\hat{A}_{n,0} = \hat{a}_n = \sum_{k=0}^{n} B_{n,k}\ \bar{\hat{a}}_k = \sum_{k=0}^{n} B_{n,k}\ \frac{\hat{B}_{n+1,1}}{n+1}.$$

The differentiation of the polynomials corresponds to right multiplication by $[IgP]$; the shift in rows, to left multiplication by the shift matrix $[S]$ with all ones on the first superdiagonal and zeros elsewhere; and the division by $n+1$, to multiplication on the left by a diagonal matrix $[Dg]$ with the diagonal $(1, 1/2, 1/3, ...)$. Explicitly,

$$[\hat{A}B] = [\hat{A}][B] = [Dg][S][B][IgP].$$

Similarly,

$$\frac{D_x}{t}\ e^{\hat{B}(t)x} = \frac{\hat{B}(t)}{t}\ e^{\hat{B}(t)x} = \bar{A}(t)\ e^{\hat{B}(t)x}$$

$$= e^{\bar{a}.t}\ e^{\hat{B}.(x)t} = e^{(\bar{a}.+\hat{B}.(x))t} = e^{\bar{A}.(\hat{B}.(x))t}$$

and

$$\frac{D_x}{t}\ e^{\hat{B}(t)x}\ |_{x=0} = \frac{\hat{B}(t)}{t} = \bar{A}(t) = e^{\bar{a}.t} = e^{\bar{A}.(\hat{B}.(0))t}$$

In addition,

$$\frac{\hat{B}(t)}{t}\ e^{\hat{B}(t)x} = \frac{t}{B(t)}\ e^{xt}\ |_{t=\hat{B}(t)} = A(t)\ e^{xt}\ |_{t=\hat{B}(t)}$$

$$= A(\hat{B}(t))\ e^{x\hat{B}(t)}$$

and

$$\frac{D_x}{t}\ e^{\hat{B}(t)x} = \sum_{n\geq 0} \frac{\hat{B}'_{n+1}(x)}{n+1}\ \frac{t^n}{n!},$$

so

$$\frac{\hat{B}'_{n+1}(x)}{n+1} = \bar{A}_n(\hat{B}.(x)) = \bar{A}\hat{B}_n(x)$$

with the matrix rep

$$[\bar{A}\hat{B}] = [\bar{A}][\hat{B}] = [Dg[S]][\hat{B}][IgP] = [Dg][S][B]^{-1}[IgP]$$

for which the first column has the e.g.f.

$$A(\hat{B}(t)) = \frac{\hat{B}(t)}{t} = \bar{A}(t) = e^{\bar{a}.t} = e^{\bar{A}.(\hat{B}.(0))t}$$

with components

$$[\bar{A}\hat{B}]_{n,0} = \frac{\hat{B}'_{n+1}(0)}{n+1} = \frac{\hat{B}_{n+1,1}}{n+1} = \bar{a}_n = \bar{A}_n(0) = \bar{A}_n(\hat{B}.(0)).$$

This along with the conjugation formula

$$]A] = [B][\bar{A}][\hat{B}]$$

imply

$$A_{n,0} = a_n = \sum_{k=0}^{n} B_{n,k}\ \bar{a}_k = \sum_{k=0}^{n} B_{n,k}\ \frac{\hat{B}_{n+1,1}}{n+1}.$$

Conversely,

$$\bar{A}_{n,0} = \bar{a}_n = \sum_{k=0}^{n} \hat{B}_{n,k}\ a_k = \frac{\hat{B}_{n+1,1}}{n+1}.$$

Note also the matrix reps

$$[A]^{-1} = [\hat{A}] = [Dg][S][B][IgP][B]^{-1} = [Dg]S][B][IgP][\hat{B}]$$

and

$$[\bar{A}] = [Dg][S][\hat{B}][IgP][B] = [Dg][S][B]^{-1}[IgP][B].$$

This last identity together with

$$[\bar{A}] = [\hat{B}][A][B] = [B]^{-1}[A][B] = [\hat{B}][A][\hat{B}]^{-1},$$

implies

$$[A] = [B][\bar{A}][\hat{B}] = [B][\bar{A}][B]^{-1} = [B][Dg][S][\hat{B}][IgP].$$

For the Bernoulli polynomials, this becomes

$$[Ber] = [St2][Dg][S][St1][IgP].$$

Since also from above $[ber] = [St2][\overline{ber}]$, the first column of $[Dg][S][St1][IgP]$ is $[\overline{ber}]$, implying

$$\overline{ber}_n = \frac{St1_{n+1,1}}{n+1} = (-1)^n \frac{n!}{n+1},$$

consistent with the formula

$$[\bar{A}\hat{B}]_{n,0} = \frac{\hat{B}'_{n+1}(0)}{n+1} = \frac{\hat{B}_{n+1,1}}{n+1} = \bar{a}_n = \bar{A}_n(0) = \bar{A}_n(\hat{B}.(0)).$$

For the unsigned hyperbinomial polynomials, we glean from

$$[\bar{A}] = [Dg][S][\hat{B}][IgP][B] = [Dg][S][B]^{-1}[IgP][B]$$

that

$$[\widetilde{Hb}] = [Dg][S][\widetilde{Ab}][IgP][\widetilde{Idp}],$$

and, from

$$[A] = [B][Dg[S]][\hat{B}][IgP]$$

that

$$[P] = [\widetilde{Idp}][Dg][S][\widetilde{Ab}][IgP].$$

Applying to

$$[\bar{A}] = [\hat{B}][A][B]$$

$$= [\widetilde{Hb}] = [\widetilde{Ab}][P][\widetilde{Idp}]$$

$$= [A088956] = [unsigned\ A137452][A007318][signed A059297]$$

$$\Rightarrow \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 3 & 2 & 1 & \\ 16 & 9 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 9 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & -2 & 1 & \\ 0 & 3 & -6 & 1 \end{bmatrix}$$

the general formulas

$$\bar{A}_{n,0} = \bar{a}_n = \sum_{k=0}^{n} \hat{B}_{n,k}\ a_k = \frac{\hat{B}_{n+1,1}}{n+1}$$

and

$$A_{n,0} = a_n = \sum_{k=0}^{n} B_{n,k}\ \bar{a}_k = \sum_{k=0}^{n} B_{n,k}\ \frac{\hat{B}_{n+1,1}}{n+1}$$

give

$$\bar{a}_n = [\widetilde{Hb}]_{n,0} = \widetilde{hb}_n$$

$= \sum_{k=0}^{n} \widetilde{Ab}_{n,k} = \frac{(n+1)^n}{n+1} = (n+1)^{n-1} = \widetilde{Ab}_n(1)$

$= A000272(n+1)$

and

$a_n = P_n(0) = 1 = \sum_{k=0}^{n} [\widetilde{Idp}]_{n,k} \ \widetilde{hb}_k$

$= \sum_{k=0}^{n} \binom{n}{k} (-k)^{n-k} \ (k+1)^{k-1} = \sum_{k=0}^{n} \binom{n}{k} (-k)^{n-k} \ \widetilde{Ab}_k(1).$