Cycles and Heat: Hermite-Sheffer Evolution Equations

(A hodgepodge of Hermite and Heisenberg courtesy the sous-chef Sheffer)

Tom Copeland, Los Angeles, Ca., June 20, 2021

Dedicated to two brighter evolutions A.A. and K.T.

This set of notes relates the basic (Graves-Pincherle-Lie-) Heisenberg-Weyl algebra to partial differential equations--evolution equations--defining the exponential generating functions (e.g.f.s) of sequences of functions that have associated ladder ops--a raising / creation op, R, and a lowering / destruction / annihilation op, L. Such ops are an integral component of quantum theory. The probabilist's, or Chebyshev, family of Hermite polynomials, whose moments--the aerated odd double factorials of matching theory--are those of a Gaussian function, play a central role in the analysis.

The ladder ops of the Sheffer polynomial sequences, with its subgroups of Appell and binomial sequences, are developed along with other aspects of the umbral operator calculus characterizing these sequences. These ladder ops form the spatial component of the Hermite-Sheffer evolution equations.

These evolution equations are then used to construct the heat / diffusion equation on the real line, deformed versions of the equation, and their solutions, presented as e.g.f.s of the Hermite polynomials composed with the cycle index polynomials of the symmetric groups S_n, a.k.a. the Stirling partition polynomials of the first kind.

At the heart of the analysis is the generalized raising op $\mathbb{R} = L + R$, which is shown to reduce to the raising op $R = x + D_x$ of the Hermite polynomials under appropriate conjugation with differential shift ops containing the Stirling partition polynomials. These two ops are characterized several ways, analytically and combinatorially.

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Ladder ops and associated evolution equations

Many important polynomial sequences ($p_n(x)$) have ladder ops--lowering/annihilation and raising/creation ops L and R--such that

$$L \, p_n(x) = n \, p_{n-1}(x) \text{ and } R \, p_n(x) = p_{n+1}(x)$$

with $L \, p_0(x) = 0.$ The commutator of the ops gives $[L, R] = 1$; i.e.,

$$[L, R] \, p_n(x) = (LR - RL) \, p_n(x) = p_n(x).$$

For operators that satisfy the commutator relations $[X, [X, Y]] = [Y, [Y, X]] = 0,$ the Baker-Campbell-Hausdorff-Dynkin expansion reduces to

$$e^{tX} e^{tY} = e^{t(X+Y) + \frac{t^2}{2}[X,Y]},$$

so the operator disentangling relationship (ODR)

$$e^{tR} e^{tL} = e^{t(R+L) + \frac{t^2}{2}[R,L]} = e^{\frac{-t^2}{2}} \, e^{t(R+L)}$$

holds, in which a Gaussian function appears.

For $p_0(x) = 1$ with the umbral notation and maneuver $p.(x)^n = p_n(x)$,

$$e^{tR} e^{tL} \, 1 = e^{tR} \, 1 = e^{t \, p.(x)} = e^{\frac{-t^2}{2}} \, e^{t(R+L)} \, 1$$

so, with the moments $h_n \, h_n$ of the Gaussian function defined by $e^{\frac{t^2}{2}} = e^{h.t}$,

$$e^{\frac{t^2}{2}} \, e^{t \, p.(x)} = e^{t \, h.} \, e^{t \, p.(x)} = e^{t \, (h.+p.(x))} = e^{t \, H.(p.(x))} = e^{t(R+L)} \, 1,$$

with the probabilist's, or Chebyshev Hermite polynomials

$$H_n(x) = (h. + x)^n = \sum_{k=0}^{n} \binom{n}{k} h_k \, x^{n-k},$$

and we can identify

$$(L + R)^n \, 1 = (h. + p.(x))^n = H_n(p.(x)) = \widetilde{H}_n(x) \; ;$$

that is,

$L + R$ is the raising op for the polynomial sequence $\widetilde{H}_n(x) = H_n(p.(x))$. (See a later section for a check / illustration of the action of $L + R$.)

Taking the derivative w.r.t. t of the exponential relation

$$e^{\frac{t^2}{2}} e^{t \, p.(x)} = e^{t(R+L)} \, 1$$

gives the evolution equation

$$D_t \, \widetilde{H}(x,t) = (R+L) \, \widetilde{H}(x,t),$$

with the solution

$$\widetilde{H}(x,t) = e^{\frac{t^2}{2}} \, e^{t \, p.(x)} = e^{h.t} \, e^{t \, p.(x)} = e^{t(h.+p.(x))} = e^{t \, H.(p.(x))} = e^{t \widetilde{H}.(x)}.$$

For $p_n(x) = x^n$, the fundamental Sheffer sequence of polynomials with $L = D_x$ and $R = x$, this evolution equation reduces to

$$D_t \, H(x,t) = (x + D_x) \, H(x,t)$$

with one solution

$$H(x,t) = e^{\frac{t^2}{2}} \, e^{tx} = e^{h.t} \, e^{tx} = e^{t(h.+x)} = e^{t \, H.(x)},$$

the exponential generating function (e.g.f.) for the family of modified Hermite polynomials $H_n(x)$ (OEIS [A099174](#)), an Appell Sheffer polynomial sequence, with the raising op

$R = x + D_x$.

The form of the polynomials gives D_x as the lowering op $L = D_x$ as for any Appell sequence; i.e.,

$$D_x \, H_n(x) = D_x \, (h. + x)^n = n \, (h. + x)^{n-1} = n \, H_{n-1}(x).$$

A quick spot check of the exponential identity:

$(x + D) \, e^{-x^2/2} = 0,$

so

$e^{t(x+D)} \, e^{-x^2/2} = e^{-x^2/2},$

consistent with

$e^{t^2/2} \, e^{tx} \, e^{tD} \, e^{-x^2/2} = e^{t^2/2} \, e^{tx} \, e^{-(x+t)^2/2} = e^{-x^2/2}.$

From the previous post on the Hermite polynomials, you can see that diverse properties of the Gaussian function/distribution follow from this operator relation. Many can be generalized. For example, the raising op formula

$$(x + D) \, H_n(x) = H_{n+1}(x)$$

gives an instance

$$H_{n+1}(x) = x \, H_n(x) - n \, H_{n-1}(x)$$

of the general recurrence relation that is a sufficient and necessary condition for a sequence to be a set of orthonormal polynomials on the real line w.r.t. a weight function/distribution (see the discussion surrounding Favard's theorem in Operator Theory: A Comprehensive Course in Analysis, Part 4 by Barry Simon).

The Hermite polynomials are an iconic instance of an Appell Sheffer polynomial sequence, another famous example being the Bernoulli polynomials. The general Sheffer polynomial sequences, all related to invertible lower triangular matrices, can also be characterized by ladder ops, which we will now derive beginning with two subgroups--the Appell and the binomial Sheffer sequences. The semidirect product of these two gives the general Sheffer sequence

But first note that there are different solutions to the Hermite evolution equation that are not necessarily the e.g.f. for a Sheffer sequence, solutions with different initial conditions. For example, from the disentangling formula with $f(x) = e^{a.x} \, f(x) = e^{a.x}$,

$$e^{t(x+D_x)} \, f(x) = e^{\frac{t^2}{2}} \, e^{tx} \, e^{tD_x} \, f(x) = e^{\frac{t^2}{2}} \, e^{tx} \, f(x+t) = e^{tH.(x)} \, f(x+t)$$

$$= e^{h. \, t} \, e^{tx} e^{a.(x+t)} = e^{(h.+a.+x)t} \, e^{a.x} = e^{t \, H.(A.(x))} \, f(x) = e^{t \, A.(H.(x))} \, f(x),$$

so the solution for the evolution equation

$$D_t \, \tilde{f}(x,t) = (x + D_x) \, \tilde{f}(x,t),$$

with the initial condition an arbitrary entire analytic function $\tilde{f}(x,0) = f(x) = e^{a.x}$ is

$$\tilde{f}(x,t) = e^{\frac{t^2}{2}} \, e^{tx} \, f(x+t) = e^{tH.(x)} \, f(x+t).$$

Appell Sheffer Sequences

Appell Sheffer sequences have an e.g.f. of the form

$$e^{A.(x)t} = A(t)\ e^{xt} = e^{a.t}\ e^{xt} = e(a. + x)t$$

with $A(0) = a_0 = 1$. The ladder ops are

$$L = D_x$$

and

$$R = (x + \frac{A'(D_x)}{A(D_x)})\ .$$

The raising op can be derived through simple differentiation of the e.g.f.

$$D_t\ A(t)\ e^{xt} = x\ A(t)\ e^{xt} + A'(t)\ e^{xt} = (x + \frac{A'(t)}{A(t)})\ A(t)e^{xt}$$

$$= D_t\ e^{A.(x)t} = A.(x)\ e^{A.(x)t} = R\ e^{A.(x)t},$$

so for Appell sequences (warning: necessary that $A(0) = a_0 = 1$),

$$R = x + \frac{A'(D_x)}{A(D_x)} = x + D_{t=D_x} \ln[A(t)] = D_{t=D_x} \ln[A(t)\ e^{xt}].$$

Note this is related to the cumulant expansion formula of OEIS A127671 and the logarithmic polynomials of A263634. (More on the op in previous posts on Appell sequences, e.g,, "The creation / raising operators for Appell sequences" and "The Pincherle derivative and the Appell raising operator.")

The lowering op follows simply from

$$D_x\ A_n(x) = D_x\ (a. + x)^n = n\ (a. + x)^{n-1} = n\ A_{n-1}(x).$$

A third way of generating an Appell sequence

in addition to via the e.g.f.

$$A(t) \; e^{xt} = e^{a.t} \; e^{xt} = e^{(a.+x)t} = e^{A.(x)t}$$

or the associated raising op

$$A_n(x) = R \; A_{n-1}(x) = R^n \; 1$$

is through the differential operation

$$A(D_x) \; x^n = e^{a.D_x} \; x^n = (a. + x)^n = A_n(x).$$

Every Sheffer sequence $S_n(x)$ has an umbral compositional inverse (UCI) defined by

$$S_n(\hat{S}.(x)) = \hat{S}_n(S.(x)) = x^n \; .$$

With $\hat{A}(t) = e^{\hat{a}.t} = \frac{1}{A(t)}$,

$$x^n = \frac{1}{A(D_x)} \; A(D_x) \; x^n = \frac{1}{A(D_x)} \; A_n(x) = e^{\hat{a}D_x} \; A_n(x)$$

$$= A_n(\hat{a}. + x) = A_n(\hat{A}.(x)) = (a. + \hat{a}. + x)^n$$

so

$$\hat{A}_n(x) = \frac{1}{A(D)} \; x^n = \hat{A}(D) \; x^n = e^{\hat{a}.t} = (\hat{a}_. + x)^n$$

is the UCI of the sequence $A_n(x)$ and vice versa. Note with $x = 0$, this gives the Kronecker delta relation

$$(a. + \hat{a}.)^n = \delta_n \; .$$

The umbral inverse allows another derivation of the raising op for an Appell sequence via operator conjugation of the raising op x of the power sequence x^n:

$$A(D) \ x \ \hat{A}(D) \ A_n(x) = A(D) \ x \ A_n(\hat{A}.(x)) = A(D) \ x \ x^{n+1} = A_{n+1}(x),$$

so

$$R = A(D) \ x \ \hat{A}(D) = A(D) \ x \ (A(D))^{-1}$$

$$= e^{a.D} \ x \ e^{\hat{a}.D} = e^{a.D} \ x \ \frac{1}{e^{a.D}}.$$

Some potential confusion could be avoided by introducing an umbral evaluation signifier $\langle \cdots \rangle$ to stress that in general

$$\hat{A}(t) = \frac{1}{e^{a.t}} = \frac{1}{\langle e^{a.t} \rangle} = \frac{1}{A(t)} \neq \langle \frac{1}{e^{a.t}} \rangle = e^{-a.t} = A(-t) \ .$$

The two expressions for the raising op for an Appell sequence can be related via the Pincherle derivative as shown in previous posts.

Note that the raising ops for the Appell sequence $A_n(x)$ and its UCI $\hat{A}_n(x)$,

$$R = D_t \ \ln[e^{xt} A(t)] \ |_{t=D_x} = x + \frac{A'(t)}{A(t)}$$

and

$$\hat{R} = D_t \ \ln[e^{xt} \hat{A}(t)] \ |_{t=D_x} = D_t \ \ln[e^{xt}/A(t)] \ |_{t=D_x} = x - \frac{A'(t)}{A(t)} \ ,$$

differ only by the sign of the derivative series component, but this is enough to distinguish the Bernoulli numbers sequence from its UCI series, the integer reciprocals $1/(n+1)$.

If $p_n(x) = A_n(x)$, an Appell sequence, multiplying on the left by $\hat{A}(D_x)$ reduces the general evolution equation to that for the Hermite polynomials:

The evolution equation again is

$$D_t \ \widetilde{H}(x,t) = (R+L) \ \widetilde{H}(x,t),$$

with the solution

$$\widetilde{H}(x,t) = e^{\frac{t^2}{2}} \ e^{t \ p.(x)} = e^{h.t} \ e^{t \ p.(x)} = e^{t(h.+p.(x))} = e^{t \ H.(p.(x))} = e^{t\widetilde{H}.(x)}.$$

so, with $p_n(x) = A_n(x)$,

$$\hat{A}(D_x)\ (R+L)\ \widetilde{H}(x,t) = \hat{A}(D_x)\ D_t\ \widetilde{H}(x,t)$$

$$= D_t\ \hat{A}\ (D_x)\widetilde{H}(x,t) = D_t\ \hat{A}(D_x)\ e^{H.(A.(x))\ t} = D_t\ e^{H.(A.(\hat{A}.(x)))t}$$

$$= D_t\ e^{H.(x)t} = D_t\ H(x,t) = (x+D)\ H(x,t).$$

Alternatively, for Appell sequences $p_n(x) = A_n(x)$,

$$R+L = A(D_x)\ x\hat{A}(D_x) + D_x = A(D_x)\ (x+D_x)\ A(D_x)^{-1}$$

and

$$\widetilde{H}(x,t) = e^{\frac{t^2}{2}}\ e^{t\ p.(x)} = A(D_x)\ e^{h.t}\ e^{tx} = A(D_x)H(x,t),$$

so

$$(R+L)\ \widetilde{H}(x,t) = A(D_x)\ (x+D_x)\ A(D_x)^{-1}\ A(D_x)H(x,t)$$

$$= A(D_x)\ (x+D_x)\ H(x,t),$$

a typical conjugation relationship as found in linear algebra--not surprising as this could all be couched in terms of lower triangular Pascal-type matrices..

We can use the UCI relationship to corroborate our exponential identity

$$e^{tR}e^{tL}\ 1 = e^{tR}\ 1 = e^{t\ p.(x)} = e^{\frac{-t^2}{2}}\ e^{t(R+L)}\ 1.$$

The reciprocal of $e^{\frac{t^2}{2}} = e^{h.t}$ is $e^{\frac{-t^2}{2}} = e^{\hat{h}t}$, implying $(h. + \hat{h}.)^n = \delta_n$, so

$$e^{\frac{-t^2}{2}}\ e^{t(R+L)}\ 1 = e^{\frac{-t^2}{2}}\ e^{t\ H.(p.(x))}$$

$$= e^{t\hat{h}.}e^{t(h.+p.(x))} = e^{(\hat{h}.+h.)t}\ e^{p.(x)t} = e^{p.(x)t}.$$

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Binomial Sheffer Sequences

Binomial Sheffer sequences, e.g., the Stirling polynomials of the first and second kinds and the Lah polynomials, have an e.g.f. of the form

$$e^{t\ B.(x)} = e^{x\ B(t)}\ ,$$

where $B(t) = e^{b.t}$, $B(0) = b_o = 0$, and $B'(0) \neq 0$. With $B^{(-1)}(t)$ the compositional inverse of the function $B(t)$ analytic at the origin, the ladder ops are

$$L = B^{(-1)}(D_x)$$

and

$$R = x\ g(D_x) = x \frac{1}{(B^{(-1)})'(D_x)}\ .$$

To validate the ladder ops, let $(w, z) = (B(z), B^{(-1)}(w))$. Then

$$L\ p_n(x) = L\ D^n_{t=0}\ e^{tp.(x)} = D^n_{t=0}\ B^{(-1)}(D_x)\ e^{tp.(x)}$$

$$= D^n_{t=0}\ B^{(-1)}(D_x)\ e^{xB(t)} = D^n_{t=0}\ B^{(-1)}(B(t))\ e^{xB(t)}$$

$$= D^n_{t=0}\ t\ e^{xB(t)} = D^n_{t=0}\ \sum_{k \geq 0}(k+1)\ p_k(x)\frac{t^{k+1}}{(k+1)!}$$

$$= n\ p_{n-1}(x).$$

The binomial Sheffer sequences are compositional polynomials that may be generated via iterated derivatives.

Consider the action of the infinitesimal generator

$$g(w)\ \frac{d}{dw} = \frac{d}{dB^{(-1)}(w)} = \frac{d}{dz}$$

on a function $f(w)$ analytic in a neighborhood about the origin

$$(g(w) \frac{d}{dw})^n f(w) = \frac{d^n}{dz^n} f(B(z)).$$

In particular,

$$(g(w) D_w)^n e^{u\,w} = D_z^n e^{u\,B(z)} = D_z^n e^{B.(u)z} = B.^n(u) e^{B.(u)z} = \sum_{k \geq 0} B_{n+k}(u) \frac{z^k}{k!}$$

and, since $w = B(z)$ evaluated at $z = 0$ gives $w = 0$,

$$(g(w) D_w)^n e^{u\,w} \mid_{w=0} = B_n(u),$$

so the coefficients of the polynomial are determined by

$$k! \, B_{n,k} = D_{u=0}^k B_n(u) = (g(w) D_w)^n w^k \mid_{w=0}.$$

Now note

$$u \, g(D_u) \sum_{k \geq 0} B_{n+k}(u) \frac{z^k}{k!}$$

$$= u \, g(D_u) \, B.^n(u) \, e^{p.(u)z} = u \, g(D_u) \, (g(w) D_w)^n \, e^{u\,w}$$

$$= (g(w) D_w)^n \, u \, g(D_u) \, e^{u\,w} = (g(w) D_w)^n \, g(w) \, D_w \, e^{u\,w}$$

$$= (g(w) D_w)^{n+1} \, e^{u\,w} = B.^{n+1}(u) \, e^{B.(u)z}$$

$$= \sum_{k \geq 0} B_{n+k+1}(u) \frac{z^k}{k!},$$

so, with a change of variables,

$$x \, g(D_x) \, B_n(x) = B_{n+1}(x) = R \, B_n(x),$$

and

$$R = x \, g(D_x) = x \, \frac{1}{B^{(-1)'}(D_x)}$$

is indeed the raising op for the binomial Sheffer sequence $B_n(x)$ with e.g.f. $e^{xB(t)}$.

Then for this sequence,

$$R + L = x\,\frac{1}{B^{(-1)\prime}(D_x)} + B^{(-1)}(D_x) = [x + \frac{D_{t=D_x}}{2}(B^{(-1)}(t))^2]\,\frac{1}{B^{(-1)\prime}(D_x)}.$$

Beware: the prime denoting differentiation of the inverse function in the denominator is easy to miss/forget, so to stress,

$$\frac{1}{B^{(-1)\prime}(D_x)} = \frac{1}{\frac{dB^{(-1)}(D_x)}{dD_x}} = \frac{1}{\frac{dB^{(-1)}(t)}{dt}}\Big|_{t=D_x}.$$

As noted above the general form of the e.g.f. of a binomial Sheffer sequence is

$$e^{tB.(x)} = e^{xB(t)}.$$

For the associated umbral compositional inverse sequence, the e.g.f. is

$$e^{t\hat{B}.(x)} = e^{x\hat{B}(t)} = e^{xB^{(-1)}(t)}.$$

This is verified by

$$e^{tB.(\hat{B}.(x))} = e^{B(t)\hat{B}.(x)} = e^{\hat{B}(B(t))x} = e^{B^{-1}(B(t))x} = e^{tx},$$

so $B_n(\hat{B}.(x)) = x^n$, and the converse holds $\hat{B}_n(B.(x)) = x^n$ since the functional compositional inversion commutes in a neighborhood about the origin.

If $p_n(x) = B_n(x)$, then operating on the general evolution equation from the left by

$$e^{\hat{B}.(y)D_{x=0}}\Big|_{y=x} = e^{yB^{(-1)}(D_{x=0})}\Big|_{y=x} = e^{(\hat{B}.(y)-y)D_x}\Big|_{y=x}$$

reduces it to the Hermite evolution equation since

$$e^{\hat{B}.(y)D_{x=0}}\Big|_{y=x}\,e^{tH.(B.(x))} = e^{tH.(B.(\hat{B}.(x)))} = e^{tH.(x)}.$$

Note the self-convolution property for binomial Sheffer sequences follows from some simple umbral mojo:

$$e^{B.(x+y)t} = e^{(x+y)B(t)} = e^{xB(t)} \, e^{yB(t)} = e^{B.(x)t} \, e^{B.(y)t} = e^{(B.(x)+B.(y))t} \ ,$$

or, equivalently,

$$B_n(x+y) = (B.(x) + B.(y))^n = \sum_{k=0}^{n} \binom{n}{k} \, B_k(y) \, B_{n-k}(x) \ .$$

An aside:

With D regarded as a partial derivative as elsewhere in this pdf, two classic evolution equations associated to compositional inversion are the inviscid Burgers-Hopf equation (A086810), related also to the KdV equation and hierarchy,

$$D_t \, U(x,t) = -U(x,t) \, D_x \, U(x,t)$$

with the solution

$$U(x.t) = \frac{x - A(x,t)}{t}$$

and the generic Lie flow equation (A145271)

$$D_t \, W(x,t) = g(x) \, D_x \, W(x,t)$$

with the solution

$$e^{tg(x)D_x} \, W(x) = W(f^{(-1)}(f(x)+t))$$

Letting $W(x) = e^{xu}$ links this equation to the binomial Sheffer sequences directly. Evaluating $e^{u \, f^{(-1)}(f(x)+t))}$ gives the e.g.f. for a binomial Sheffer sequence.

Because of the nonlinear term, the inviscid Burgers equation can be translated into the convolutional properties of Lagrange partition polynomials for compositional inversion of formal power series and conversely. See A145271 and links in the cross references section there for several sequences of Lagrange partition polynomials for compositional inversion.

General Sheffer Sequences

General Sheffer sequences, e.g., the associated Laguerre polynomials of order other than -1, have an e.g.f. of the composite form

$$A(t) \; e^{xB(t)} = e^{t(a.+B.(x))} = e^{tS.(x)}$$

with $A(t) = e^{a.t}$ and $B(t) = e^{b.t}$ with $A(0) = a_0 = 1$, $B(0) = b_0 = 0$, and $B'(0) \neq 0$.

I'll verify below that, with $g(D_x) = \frac{1}{(B^{(-1)})'(x)}$, the ladder ops for general Sheffer sequences are

$$L = B^{(-1)}(D_x)$$

 and

$$R = [x \; g(D_x) + \frac{A'(B^{(-1)}(D_x))}{A(B^{(-1)}(D_x))}].$$

The lowering op gives

$$L \; A(t) \; e^{xB(t)} = B^{(-1)}(D_x) \; A(t) \; e^{xB(t)} = B^{(-1)}(B(t)) \; A(t) \; e^{xB(t)}$$

$$= t \; A(t) \; e^{xB(t)} = te^{tS.(x)} = \sum_{n \geq 0} (n+1) \; S_n(x) \frac{t^{n+1}}{(n+1)!}$$

$$= \sum_{n \geq 0} n \; S_{n-1}(x) \frac{t^n}{n!} = L \; e^{tS.(x)}$$

with $S_{-1}(x) = L \; S_0(x) = L \; 1 = 0.$

Now for determining the raising op, take the derivative

$$D_t \; e^{tS.(x)} = S.(x) \; e^{tS.(x)} = R \; e^{tS.(x)}$$

$$= D_t \; A(t) \; e^{xB(t)} = [A'(t) + A(t) \; B'(t)] \; e^{xB(t)}.$$

We know from the raising op of binomial Sheffer sequences that

$$x \, g(D_x) \, A(t) \, e^{xB(t)} = A(t) \, B.(x) \, e^{tB.(x)} = A(t) \, D_t \, e^{xB(t)}$$

$$= A(t) \, B'(t) \, e^{xB(t)}.$$

In addition,

$$\frac{A'(B^{(-1)}(D_x))}{A(B^{(-1)}(D_x))} \, A(t) \, e^{xB(t)} = A'(t) \, e^{xB(t)}.$$

Combining these expressions gives

$$R \, A(t) \, e^{xB(t)} = D_t \, A(t) \, e^{xB(t)} = \left[A(t) \, B'(t) + A'(t) \right] e^{xB(t)}$$

$$= \left[x \, g(D_x) + \frac{A'(B^{(-1)}(D_x))}{A(B^{(-1)}(D_x))} \right] \, A(t) \, e^{xB(t)}.$$

Then

$$R = \left[x \, g(D_x) + \frac{A'(B^{(-1)}(D_x))}{A(B^{(-1)}(D_x))} \right].$$

Also

$$D_t \, \ln[\, A(B^{(-1)}(t)) \,] = \frac{A'(B^{(-1)}(t))}{A(B^{(-1)}(t))} \, B^{(-1)'}(t) = \frac{A'(B^{(-1)}(t))}{A(B^{(-1)}(t))} \, \frac{1}{g(t)},$$

so, alternatively,

$$R = [\, x + D_{t=D_x} \, \ln[\, A(B^{(-1)}(t)) \,] \,] \, g(D_x).$$

Quick sanity checks:

For Appell sequences, $B(t) = B^{(-1)(t)} = t$, so $g((D_x) = 1$, giving $R_A = x + \frac{A'(x)}{A(x)}$.

For binomial sequences, $A(t) = 1$, so $R_B = xg(D_x)$.

For the Sheffer sequence with e.g.f. $A(t) \, e^{xB(t)} = (1-t) \, e^{x \, ln(1-t)} = (1-t)^{x+1}$, then $L = 1 - e^{D_x}$ \ and $R = (x+1) \, (-e^{-D_x})$, and the general evolution equation gives, agreeably,

$$D_t \, e^{\frac{t^2}{2}} \, (1-t)^{x+1} = (t - \frac{1+x}{1-t}) \, e^{\frac{t^2}{2}} \, (1-t)^{x+1}$$

$$= (L+R) \, e^{\frac{t^2}{2}} \, (1-t)^{x+1} = [1 - e^{D_x} - (1+x)e^{-D_x}] \, e^{\frac{t^2}{2}} \, (1-t)^{x+1}$$

$$= (1 - (1-t) - \frac{1+x}{1-t}) \, e^{\frac{t^2}{2}} \, (1-t)^{x+1}.$$

Now for the UCI sequence $\hat{S}_n(x)$ of a general Sheffer sequence $S_n(x)$:

For a general Sheffer sequence,

$$e^{S.(x)t} = A(t) \, e^{x \, B(t)} = e^{a.t} \, e^{B.(x)t} = e^{(a.+B.(x))t} = e^{A.(B.(x))t},$$

so

$$S_n(x) = A_n(B.(x) \, .$$

The UCI is

$$\hat{S}_n(x) = \hat{B}_n(\hat{A}.(x))$$

Since

$$S_n(\hat{S}.(x)) = A_n(B.(\hat{B}.(\hat{A}.(x)))) = A_n(\hat{A}(x))) = x^n$$

and, conversely,

$$\hat{S}_n(S.(x)) = \hat{B}_n(\hat{A}.(A.(B.(x)))) = \hat{B}_n(B.(x)) = x^n \, .$$

The e.g.f. of the UCI is then

$$e^{\hat{S}.(x)t} = e^{\hat{B}.(\hat{A}.(x))t} = e^{t(\hat{B}.(x+\hat{a}.))} = e^{(x+\hat{a}.)B^{(-1)}(t)}$$

$$= e^{\hat{a}.B^{(-1)}(t)} \, e^{xB^{(-1)}(t)} = \hat{A}(B^{(-1)}(t)) \, e^{xB^{(-1)}(t)} = \frac{1}{A(B^{(-1)}(t))} \, e^{xB^{(-1)}(t)}.$$

Introduce the convenient notation for any two symbols $(: XY :)^n = X^n \, Y^n$, which distributes the exponent while preserving the order of the symbols. (Careful, don't confuse this with normal ordering--simply $: xD :^n = x^n D^n$ and $: Dx := D^n x^n$ with this notation.) Then there are essentially three useful equivalent reps for the umbral composition of a Sheffer sequence with a formal power series or function $h(x)$ analytic about the origin,

$$h(\hat{S}.(x)) = e^{\hat{S}.(x)D_y} \, h(y) \, |_{y=0} = \frac{1}{A(B^{(-1)}(D_{y=0}))} \, e^{xB^{(-1)}(D_{y=0})} \, h(y)$$

$$= e^{:\hat{S}.(x)D_{x=0}:} \, h(x) = \sum_{n\geq 0} \hat{S}_n(x) \, \frac{D_{x=0}^n}{n!} \, h(x)$$

$$= e^{:(\hat{S}.(x)-x)D_x:} \, h(x) = \sum_{n\geq 0} \frac{(\hat{S}.(x)-x)^n \, D_x^n}{n!} \, h(x) \ .$$

The last expression follows from the property of the binomial coefficients underlying the simple identity

$$x^n = (1-(1-x)^n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \sum_{k=0}^{m} \binom{m}{k} (-x)^k$$

$$= \sum_{k\geq 0} (-x)^k \sum_{m\geq 0} (-1)^m \binom{n}{m} \binom{m}{k},$$

with the natural convention $1/(-n)! = \lim_{x\to -n} 1/\Gamma(x+1) = 0$ for $n = 1,\, 2,\, 3, \dots.$, implying

$$\sum_{m\geq 0} (-1)^m \binom{n}{m} \binom{m}{k} = (-1)^k \, \delta_{n-k}.$$

Then explicitly and laboriously, for those skeptical of umbral witchcraft,

$$e^{:(\hat{S}.(x)-x)D_x:} \, x^n = \sum_{m\geq 0} \frac{(\hat{S}.(x)-x)^m \, D_x^m}{m!} \, x^n = \sum_{m\geq 0} (\hat{S}.(x)-x)^m \binom{n}{m} x^{n-m}$$

$$= \sum_{m\geq 0} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \hat{S}_k(x) \, x^{m-k} \binom{n}{m} x^{n-m}$$

$$= \sum_{k\geq 0} (-1)^k \, \hat{S}_k(x) \, x^{n-k} \sum_{m\geq 0} (-1)^m \binom{m}{k} \binom{n}{m}$$

$$= \sum_{k\geq 0} (-1)^k \, \hat{S}_k(x) \, x^{n-k} \, (-1)^k \delta_{n-k} = S_n(x) \ ,$$

so, indeed, the generalized umbral shift diff op for $q.$ commuting with x

$$e^{:q.D_x:} \, x^n = (q. + x)^n = e^{:p.(x)D_x:} \, x^n = (p.(x) + x)^n \ ,$$

at the heart of umbral witchcraft, gives

$$e^{:(\hat{S}.(x)-x)D_x:} \, x^n = (\hat{S}.(x) - x + x)^n = \hat{S}.(x)^n = \hat{S}_n(x) \ .$$

Note for reference that for q_m commuting with the derivative,

$$e^{q.D_{x=0}} \, x^n = q_n = e^{-(1-q.)D_{x=1}} \, x^n = (x - (1 - q.))^n|_{x=1} = (1 - (1 - q.)^n = (q.)^n \ .$$

The second diff op often allows Newton interpolation and Mellin transform interpolation on expressions not analytic at the origin but at the identity, e.g., for action on x^s for s not a natural integer. For q_m commuting with x x but not the derivative,

$$e^{:q.D_{x=0}:} \, x^n = q_n = e^{-:(1-q.)D_{x=1}:} \, x^n = (x - (1 - q.))^n|_{x=1} = (1 - (1 - q.)^n = (q.)^n \ .$$

In this context, the three substitution, or composition, diff ops (SDOs)

$$\frac{1}{A(B^{(-1)}(D_{x=0}))} \, e^{yB^{(-1)}(D_{x=0})} \, \Big|_{y=x} = e^{:\hat{S}.(x)D_{x=0}:} = e^{:(\hat{S}.(x)-x)D_x:}$$

are equivalent when acting on the terms of a formal power series. Then for the UCI Sheffer sequence $\hat{S}_n(x)$ for any Sheffer sequence $S_n(x)$, which has the ladder ops L and R, any of the three SDOs gives

$$e^{:\hat{S}.(x)D_{x=0}:} \, H_n(S.(x)) = H_n(S.(\hat{S}.(x))) = H_n(x).$$

Recall the associated general Sheffer evolution equation

$$D_t \, \widetilde{H}(x,t) = (R + L) \, \widetilde{H}(x,t),$$

with the solution

$$\widetilde{H}(x,t) = e^{\frac{t^2}{2}} \, e^{t \, S.(x)} = e^{h.t} \, e^{t \, S.(x)} = e^{t(h.+S.(x))} = e^{t \, H.(S.(x))} = e^{t\widetilde{H}.(x)}.$$

Then the general evolution equation is reduced to that for the Appell Hermite polynomials by acting on the equation with any of the SDOs above; explicitly,

$$e^{:\hat{S}.(x)D_{x=0}:} \, D_t \, \widetilde{H}(x,t) = D_t \, e^{:\hat{S}.(x)D_{x=0}:} \, \widetilde{H}(x,t)$$

$$= D_t \, e^{:\hat{S}.(x)D_{x=0}:} \, e^{t \, H.(S.(x))} = D_t \, e^{t \, H.(S.(\hat{S}.(x)))}$$

$$= D_t \, e^{t \, H.(x)} = D_t \, H(x,t) = (x + D_x) \, H(x.t),$$

but

$$D_t \, \widetilde{H}(x,t) = (R + L) \, \widetilde{H}(x,t)$$

and

$$e^{:S.(x)D_{x=0}:} \, H(x,t) = H(S.(x),t) = \widetilde{H}(x,t),$$

so, switching SDOs,

$$e^{:(\hat{S}.(x)-x)D_x:} \, (R + L) \, e^{:(S.(x)-x)D_x:} \, e^{t \, H.(x)} = (x + D_x)e^{t \, H.(x)}$$

$$= e^{:(\hat{S}.(x)-x)D_x:} \, (R + L) \, e^{:(S.(x)-x)D_x:} \, e^{\frac{t^2}{2}} e^{xt} = (x + D_x) \, e^{\frac{t^2}{2}} e^{xt}.$$

Removing the Gaussian,

$$e^{:(\hat{S}.(x)-x)D_x:} \, (R + L) \, e^{:(S.(x)-x)D_x:} \, e^{xt} = (x + D_x) \, e^{xt} = (x + t) \, e^{xt}.$$

Then

$$e^{-xt} \, e^{:(\hat{S}.(x)-x)D_x:} \, (R + L) \, e^{:(S.(x)-x)D_x:} \, e^{xt} = e^{-xt} \, (x + D_x) \, e^{xt} = (x + t).$$

The diff operators on either side are equivalent when acting on e^{-xz} or on each individual signed or unsigned Hermite polynomial, so they are equivalent when acting on functions expanded in terms of the orthonormal basis of Hermite polynomials or the Laplace or inverse Laplace transforms of functions. Consequently, a least over these classes of functions, we have the operator conjugation relation (OCR)

$$e^{:(\hat{S}.(x)-x)D_x:} \, (R + L) \, e^{:(S.(x)-x)D_x:} = x + D_x$$

Since

$$e^{:(\hat{S}.(x)-x)D_x:} \; e^{:(S.(x)-x)D_x:} \; f(x) = f(S.(\hat{S}.(x))) = f(x) \; ,$$

also

$$e^{:(\hat{S}.(x)-x)D_x:} \; e^{t(R+L)} \; e^{:(S.(x)-x)D_x:} = e^{t(x+D_x)} .$$

Then, by the disentangling identity,

$$e^{:(\hat{S}.(x)-x)D_x:} \; e^{tR} e^{tL} \; e^{:(S.(x)-x)D_x:} = e^{tx} e^{tD_x} ,$$

so, for $f(x) = e^{c.x}$,

$$e^{:(\hat{S}.(x)-x)D_x:} \; e^{tR} e^{tL} \; e^{:(S.(x)-x)D_x:} \; f(x)$$

$$= e^{tx} e^{tD_x} \; f(x) = e^{tx} \; f(x+t) = e^{c.(x+t)+tx} .$$

Check:

$$e^{:(\hat{S}.(x)-x)D_x:} \; e^{tR} e^{tL} \; e^{:(S.(x)-x)D_x:} \; e^{xt} = e^{:(\hat{S}.(x)-x)D_x:} \; e^{tR} e^{tL} \; e^{tS.(x)} ,$$

and

$$e^{tL} \; S_n(x) = \sum_k^n \binom{n}{k} \; t^k \; S_{n-k}(x) = (S.(x)+t)^n ,$$

so

$$e^{tL} \; e^{tS.(x)} = e^{t(S.(x)+t)} = e^{t^2} \; e^{tS.(x)} = e^{\sqrt{2}h.t} \; e^{tS.(x)} .$$

$$e^{tR} \; S_n(x) = \sum_{k \geq 0} \frac{t^k}{k!} \; S_{n+k}(x) = S.(x)^n e^{tS.(x)} .$$

$$e^{tR} \; e^{tS.(x)} = \sum_{k \geq 0} \frac{t^k}{k!} \; S.(x)^k \; e^{tS.(x)} = e^{tS.(x)} \; e^{tS.(x)} = e^{2tS.(x)}$$

and

$$e^{:(\hat{S}.(x)-x)D_x:} \; e^{tR} e^{tL} \; e^{:(S.(x)-x)D_x:} \; e^{xt} = e^{:(\hat{S}.(x)-x)D_x:} \; e^{t^2} \; e^{2tS.(x)}$$

$$= e^{t^2} \; e^{2tS.(\hat{S}(x))} = e^{t^2} \; e^{2tx} ,$$

which agrees with

$$e^{tx} \, e^{tD_x} \, e^{xt} = e^{tx} \, e^{t(x+t)} = e^{t^2} \, e^{2tx} = e^{\bar{H}.(2x)t},$$

where $e^{\bar{H}.(x)t} = e^{t^2} \, e^{tx}$, another family of Hermite polynomials

Note: For any Appell sequence,

$e^{xt} = e^{-\bar{a}.t} \, e^{\bar{A}.(x)t}$, so

$$e^{:(\hat{S}.(x)-x)D_x:} \, e^{tR} e^{tL} \, e^{:(S.(x)-x)D_x:} \, e^{\bar{A}.(x)t} = e^{\bar{a}.t} \, e^{t^2} \, e^{2tx}$$

$$= e^{t^2} \, e^{tx} e^{\bar{A}.(x)t} = e^{\bar{H}.(x)t} \, e^{\bar{A}.(x)t}$$

$$= e^{\bar{H}.(x+\bar{A}.(x))t} = e^{\bar{A}.(\bar{H}.(x)+x)t}$$

$$= e^{\bar{H}.(\bar{A}.(2x))t} = e^{\bar{A}.(\bar{H}.(2x))t}.$$

Note:

$$e^{t^2} e^{2tx} = e^{\frac{\bar{t}^2}{2}} e^{\bar{t}\bar{x}} = e^{\bar{t}H.(\bar{x})} = e^{\sqrt{2}\,tH.(\sqrt{2})\,x}$$

with $\bar{t} = t\sqrt{2}$ and $\bar{x} = x\sqrt{2}$.

Note:

$e^{-t^2} \, e^{t(R+L)} \, e^{tS.(x)} = e^{tR} \, e^{tL} \, e^{tS.(x)} = e^{t^2} \, e^{2tS.(x)}$, so

$e^{t(R+L)} \, e^{tS.(x)} = e^{3\frac{t^2}{2}} \, e^{2tS.(x)}$.

Note:

$e^{tR} \, e^{tL} \, e^{tS.(x)} = e^{t^2} \, e^{2tS.(x)}$.

Check:

For Appell sequences,

$S_n(x) = A_n(x) = (a. + x)^n$,

$$L = D_x \, ,$$

$$R = e^{a.D_x} \, x \, a^{\hat{a}.D_x} \, ,$$

$$(S.(x) - x)^n = (a. + x - x)^n = a_n \, ,$$

and

$$e^{tR} = e^{a.D_x} \, e^{tx} \, a^{\hat{a}.D_x}$$

since $\; e^{a.D_x} \, a^{\hat{a}.D_x} = 1$.

Then our general Sheffer OCR consistently reduces to

$$e^{\hat{a}.D_x} \, e^{a.D_x} \, e^{tx} \, a^{\hat{a}.D_x} \, e^{tD_x} \, e^{a.D_x} = e^{tx} \, e^{tD_x} \; .$$

Check the formula for a Sheffer UCI:

The normalized associated Laguerre polynomials $L_n^{(\alpha)}(x)$ have the e.g.f.s

$$e^{tL.^{(\alpha)}(x)} = \frac{1}{(1-t)^{\alpha+1}} e^{x \frac{t}{t-1}} = A(t) \, e^{xB(t)},$$

Then

$$B^{(-1)}(t) = \tfrac{t}{t-1} = B(t)$$

and

$$\frac{1}{A(B^{(-1)}(t))} = (1 - \tfrac{t}{t-1})^{\alpha+1} = A(t),$$

implying that the normalized associated Laguerre polynomials are self-inverse under umbral composition.

Check the self- inverse property for the associated Laguerre polynomials:

$$L_n^{(\alpha)}(x) = n! \sum_{k=0}^{n} \binom{n+\alpha}{k+\alpha} \frac{(-x)^k}{k!},$$

so

$$L_n^{(\alpha)}(L.^{(\alpha)}(x)) = n! \sum_{k=0}^{n} \binom{n+\alpha}{k+\alpha} \frac{(-L.^{(\alpha)}(x))^k}{k!} = n! \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{k+\alpha} \frac{L_k^{(\alpha)}(x)}{k!}$$

$$= n! \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{k+\alpha} \sum_{j=0}^{k} \binom{k+\alpha}{j+\alpha} \frac{(-x)^j}{j!}$$

$$= n! \sum_{k=0}^{n} (-1)^k \sum_{j=0}^{k} \binom{n+\alpha}{k+\alpha} \binom{k+\alpha}{j+\alpha} \frac{(-x)^j}{j!}$$

$$= n! \sum_{j=0}^{\infty} \frac{(-x)^j}{j!} \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{k+\alpha} \binom{k+\alpha}{j+\alpha},$$

and

$$Sum = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{k+\alpha} \binom{k+\alpha}{j+\alpha}$$

$$= \frac{(n+\alpha)!}{(j+\alpha)!} \sum_{k=0}^{n} (-1)^k \frac{1}{(n-k)!\,(k-j)!}$$

$$= \frac{(n+\alpha)!}{(j+\alpha)!} (-1)^n \sum_{m=0}^{n} (-1)^m \frac{1}{m!\,(n-m-j)!}$$

$$= \binom{n+\alpha}{j+\alpha} (-1)^n \sum_{m=0}^{n} (-1)^m \binom{n-j}{m}$$

$$= \binom{n+\alpha}{j+\alpha} (-1)^n (1-1)^{n-j} = (-1)^n \delta_{n-j}$$

with $m = n - k$ $m = n - k$, so

$$L_n^{(\alpha)}(L.^{(\alpha)}(x)) = n! \sum_{j=0}^{\infty} \frac{(-x)^j}{j!} (-1)^n \delta_{n-j} = x^n.$$

Check the Sheffer op formalism and the evolution equation for a particular binomial Sheffer sequence:

For the binomial Sheffer sequence the signed Lah polynomials (OEIS A111596),

$$p_n(x) = Lah_n(x) = L_n^{-1}(x)$$

with e.g.f.

$$e^{tLah.(x)} = e^{x\frac{t}{t-1}},$$

so

$$B(t) = B^{(-1)}(t) = \frac{t}{t-1}$$

and

$$B^{(-1)'}(t) = \frac{1}{t-1} - \frac{t}{(t-1)^2} = \frac{-1}{(t-1)^2},$$

so

$$R + L = x \, \frac{1}{B^{(-1)'}(D_x)} + B^{(-1)}(D_x) = -x(1 - D_x)^2 + \frac{D_x}{D_x - 1}$$

$$= -x + 2xD_x - xD_x^2 - D_x - D_x^2 - D_x^3 - \dots$$

$$= -[x + (1 - 2x)\, D_x + (1 + x)D_x^2 + D_x^3 + D_x^4 + \dots]$$

$$= [x(D_x - 1)^3 + D_x] \, \frac{1}{D_x - 1} = [x(D_x^3 - 3D_x^2 + 3D_x - 1) + D_x] \, \frac{1}{D_x - 1}.$$

Consistently,

$$D_t \, e^{\frac{t^2}{2}} \, e^{t \, Lah.(x)} = [t + Lah.(x)] \, e^{\frac{t^2}{2}} \, e^{t \, Lah.(x)}$$

$$= (L + R) \, e^{\frac{t^2}{2}} \, e^{t \, Lah.(x)} = e^{\frac{t^2}{2}} \sum_{n \geq 0} [n \, Lah_{n-1}(x) + Lah_{n+1}(x)] \, \frac{t^n}{n!}.$$

Alternatively,

$$D_t \, e^{\frac{t^2}{2}} \, e^{x\frac{t}{t-1}} = [t - x\frac{1}{(1-t)^2}] \, e^{\frac{t^2}{2}} \, e^{x\frac{t}{t-1}},$$

and, consistently,

$$(R+L)\, e^{\frac{t^2}{2}}\, e^{x\,\frac{t}{t-1}} = [-x(1-D_x)^2 + \frac{D_x}{D_x-1}]\, e^{\frac{t^2}{2}}\, e^{x\,\frac{t}{t-1}}$$

$$= [-x(1-\frac{t}{t-1})^2 + \frac{\frac{t}{t-1}}{\frac{t}{t-1}-1}]\, e^{\frac{t^2}{2}}\, e^{x\,\frac{t}{t-1}}$$

$$= [-x\frac{1}{(t-1)^2} + t]\, e^{\frac{t^2}{2}}\, e^{x\,\frac{t}{t-1}}.$$

Illustrate the ladder ops for the normalized Laguerre polynomials ($\alpha = 0$ above):

The normalized Laguerre polynomials $L_n(x)$, an example of orthogonal, general Sheffer sequence, have the e.g.f.

$$e^{tL.(x)} = \frac{1}{1-t}\, e^{x\,\frac{t}{t-1}},$$

the lowering op

$$L = B^{(-1)}(D_x) = B(D_x) = \frac{D_x}{D_x-1} = -D_x - D_x^2 - D_x^3 + ...$$

and since

$$A(B^{(-1)}(t)) = \frac{1}{1-\frac{t}{t-1}} = 1-t \text{ and } g(t) = \frac{1}{D_t\frac{t}{t-1}} = -(1-t)^2,$$

the raising op is

$$R = [\, x + D_{t=D_x}\ \ln[\ A(B^{(-1)}(t))\,]\,]\, g(D_x) = [x + D_{t=D_x}\ \ln(1-t)\,]\,[-(1-D_x)^2]$$

$$= (x - \frac{1}{1-D_x})\,[-(1-D_x)^2] = -x(1-D_x)^2 + (1-D_x) = 1 - x + (2x-1)D_x - xD_x^2.$$

Then the first few Laguerre polynomials are

$$L_0(x) = 1,$$

$$L_1(x) = R\,1 = 1 - x\,.$$

$$L_2(x) = R\ (1-x) = (1-x)^2 + 1 - 2x = 2 - 4x + x^2 \ .$$

$$L_3(x) = (1-x)^3 + (1-x)(1-2x) + (2x-1)(-4+2x) - 2x = 6 - 18x + 9x^2 - x^3 \ .$$

Check that the lowering op is correct; that is, that

$$-(D_x + D_x^2 + ... + D_x^n)\ L_n(x) = n\ L_{n-1}(x)\ .$$

Then

$$L + R = \frac{D_x}{D_x - 1} + (x - \frac{1}{1 - D_x})\ [-(1 - D_x)^2]$$

$$= 1 - x - 2(1-x)D_x - (1+x)D_x^2 - D_x^3 - D_x^4 - ... \ .$$

A Deformed Heat Equation and Appell and Binomial Sheffer Sequences

Binomial Sheffer sequences are intimately related to Appell sequences in a distinguished indeterminate.

The partition polynomials $P_n(a_1, a_2, a_3, ...; x)$ associated with functional composition are binomial Sheffer sequences in the variable x but can be treated as Appell polynomials in the distinguished indeterminant a_1 with $x = 1$. Three examples are the Stirling partition polynomials of the second kind, a.k.a. the Faa di Bruno polynomials; the Stirling partition polynomials of the first kind, a.k.a. the cycle index polynomials of the symmetric groups S_n; and the Lah partition polynomials, an umbralization of the normalized, unsigned Laguerre polynomials of order -1.

Their e.g.f.s are, respectively,

$$e^{t \; St2.(c_1,...;z)} = e^{z \; (e^{c.t}-1)} = \exp[\; z \sum_{n \geq 1} c_n \frac{t^n}{n!} \;] = e^{zt \; c_1} \; \exp[\; z \sum_{n \geq 2} c_n \frac{t^n}{n!} \;]$$

$$e^{t \; St1.(c_1,...;z)} = e^{-z \; \ln(1-c.t)} = \exp[\; z \sum_{n \geq 1} c_n \frac{t^n}{n} \;] = e^{zt \; c_1} \; \exp[\; z \sum_{n \geq 2} c_n \frac{t^n}{n} \;]$$

$$e^{t \; Lah.(c_1,...;z)} = e^{z \; \frac{c.t}{1-c.t}} = \exp[\; z \sum_{n \geq 1} c_n \; t^n \;] = e^{zt \; c_1} \; \exp[\; z \sum_{n \geq 2} c_n \; t^n \;].$$

Let's focus on the Stirling partition polynomials of the first kind. From the pdf in my post Lagrange à la Lah, the first few are

$St1_0 = 1$,

$St1_1(c_1; z) = c_1 z = St1_1(c_1 z; 1)$,

$St1_2(c_1, c_2; z) = c_2 z + c_1^2 z^2 = St1_2(c_1 z, c_2 z; 1)$,

$St1_3(c_1, c_2, c_3; z) = 2c_3 z + 3c_1 c_2 z^2 + c_1^3 z^3 = St1_3(c_1 z, c_2 z, c_3 z; 1)$,
$St1_4(c_1, c_2, c_3, c_4; z) = 6c_4 z + (8c_1 c_3 + 3c_2^2)z^2 + 8c_1^2 c_2 z^3 + c_1^4 z^4.$

With $z = 1$ and $c_1 = x$,

$$L \; St1_n(x, c_2, ..., c_n; 1) = D_x \; St1_n(x, c_2, ..., c_n; 1) = n \; St1_{n-1}(x, c_2, ..., c_{n-1}; 1)$$

and

$$R = D_{t=D_x} \; \ln[\; e^{t \; x} \; \exp[\sum_{n \geq 2} c_n \frac{t^n}{n} \;]\;] = x + \sum_{n \geq 1} c_{n+1} \; D_x^n$$

$$= x + c. \sum_{n \geq 1} (c.D_x)^n = x + \frac{c.}{1 - c.D_x}$$

$$= D_{t=D_x} \; \ln[\exp[-\ln(1 - c.t)] = D_{t=D_x} \; [-\ln(1 - c.t)].$$

(Similar simple expressions apply to the other partition polynomial in umbral notation.)
Consequently, the solution to the evolution equation for this Appell sequence

$$D_t\ \widetilde{H}(x,t) = (R+L)\ \widetilde{H}(x,t) = [\ x + (1+c_2)D_x + \sum_{n\geq 2} c_{n+1}\ D_x^n\ ]\ \widetilde{H}(x,t),$$

is

$$\widetilde{H}(x,t) = e^{\frac{t^2}{2}}\ e^{xt}\ \exp[\ \sum_{n\geq 2} c_n\ \frac{t^n}{n}\ ].$$

$$= e^{h.t}\ e^{xt}\ e^{t\ St1.(0,c_2,...;1)} = e^{t\ (h.+St1.(0,c_2,...;1)+x)}$$

$$= e^{t\ H.(St1.(x,c_2,...;1))} = e^{t\ St1.(H.(x),c_2,...;1)}.$$

This equation can be converted into the one for the Hermite e.g.f., as demonstrated in the sections above.

Given the Appell polynomial sequence defined by

$$e^{x\ A(t)} = e^{t\ A.(x)} = e^{t(x+a.)} = e^{a.x}\ e^{tx} = e^{t\ St1.(0,c_2,...;1)}e^{tx}$$

$$= e^{t\ (St1.(0,c_2,...;1)+x)} = e^{t\ St1.(x,c_2....;1)} = \exp[\ -\ln(1-\bar{c}.t)\ ],$$

with $\bar{c}_1 = x$ and $\bar{c}_n = c_n$ otherwise,

the e.g.f of the UCI is

$$e^{x\ \hat{A}(t)} = e^{t\ \hat{A}.(x)} = e^{t(x+\hat{a}.)} = e^{\hat{a}.t}\ e^{tx}$$

$$= \frac{1}{e^{t\ St1.(0,c_2,...;1)}}\ e^{tx} = \frac{1}{e^{t\ St1.(0,c_2,...;1)}\ e^{tx}}\ e^{2tx} = \frac{1}{e^{t\ St1.(x,c_2,...;1)}}\ e^{2tx}$$

$$= \frac{1}{e^{-\ln(1-\bar{c}.t)}}\ e^{2tx} = e^{\ln(1-\bar{c}.t)}\ e^{2tx}$$

$$= e^{t\ St1.(0,-c_2,-c_3,...;1)}e^{tx} = e^{t\ St1.(x,-c_2,-c_3,...;1)} = e^{-\ln(1-\beta.t)},$$

with $\beta_1 = \bar{c}_1 = x$ and $\beta_n = -\bar{c}_n = -c_n$ otherwise.

The UCI sequence is given by

$$\hat{A}_n(x) = (x + a.)^n = [x + St1.(0, -c_2, -c_3, ...; 1)]^n = St1_n(x, -c_2, -c_3, ...; 1)$$

and is generated by

$$e^{\hat{a}. \ D_x} \ x^n = \hat{A}.(x) = e^{St1.(0, -c_2, -c_3, ...; 1) \ D_x}.$$

Recalling the definition of an Appell UCI pair,

$$x^n = e^{a. \ D_x} \ \frac{1}{e^{a. \ D_x}} \ x^n = e^{a. \ D_x} \ e^{\hat{a}. \ D_x} \ x^n$$

$$= (a. + \hat{a}. + x)^n = A_n(\hat{A}.(x)) = \hat{A}_n(A.(x)),$$

so

$$e^{St1.(0, -c_2, -c_3, ...; 1) \ D_x} \ St1_n(x, c_2, c_3, ..., c_n; 1)$$

$$= St1_n[St1_n(x, -c_2, -c_3, ..; 1), c_2, c_3, ..c_n; 1] = x^n$$

and

$$e^{St1.(0, -c_2, -c_3, ...; 1) \ D_x} \ \widetilde{H}(x, t)$$

$$= e^{St1.(0, -c_2, -c_3, ...; 1) \ D_X} \ e^{\frac{t^2}{2}} \ e^{xt} \ e^{t \ St1.(0, c_2, ...; 1)}$$

$$= e^{St1.(0, -c_2, -c_3, ...; 1) \ D_x} \ e^{\frac{t^2}{2}} \ e^{t \ St1.(x, c_2, ...; 1)} = e^{\frac{t^2}{2}} \ e^{xt}.$$

Consequently,

$$e^{St1.(0, -c_2, -c_3, ...; 1) \ D_x} \ D_t \ \widetilde{H}(x, t) = D_t \ H(t, x)$$

$$= (D + x) \ H(x, t) = e^{St1.(0, -c_2, -c_3, ...; 1) \ D_x} \ (L + R) \ \widetilde{H}(x, t)$$

$$= e^{St1.(0, -c_2, -c_3, ...; 1) \ D_x} \ (L + R) \ e^{St1.(0, c_2, c_3, ...; 1) \ D_x} H(x, t),$$

Giving, as also shown in previous sections in the general case, for the Appell sequence operators

$$e^{St1.(0,-c_2,-c_3,...;1)\ D_x}\ (L+R)\ e^{St1.(0,c_2,c_3,...;1)\ D_x} = D + x,$$

or

$$L + R = e^{St1.(0,c_2,c_3,...;1)\ D_x}\ (D+x)\ e^{St1.(0,-c_2,-c_3,...;1)\ D_x},$$

consistent with the other derivations above of the raising op for an Appell sequence.

From the Appell formalism,

$$e^{\frac{D_x^2}{2}}\ \exp\Big[\sum_{k\geq 2} c_k\ \frac{D_x^k}{k}\Big]\ x^n$$

$$= e^{h.D_x}\ e^{St1.(0,c_2,...;1)D_x}\ x^n = e^{(h.+St1.(0,c_2,...;1))D_x}\ x^n$$

$$= (h. + St1.(0,c_2,...;1) + x)^n = H_n(St1.(x,c_2,...;1)) = St1.(H(x),c_2,...;1)$$

$$= (St1.(h,c_2,...;1) + x)^n = (H.(St1.(0,c_2,...;1)) + x)^n\ .$$

Letting $t \geq 0$ be time,

$$D_t\ e^{t\frac{D_x^2}{2}}\ \exp\Big[\ t\sum_{n\geq 2} c_n\ \frac{D_x^n}{n}\Big]\ f(x)$$

$$= \Big[\ \frac{D_x^2}{2} + \sum_{n\geq 2} c_n\ \frac{D_x^n}{n}\Big]\ e^{t\frac{D_x^2}{2}}\ \exp\Big[\ t\sum_{n\geq 2} c_n\ \frac{D_x^n}{n}\Big]\ f(x),$$

leads to the deformed heat / diffusion equation

$$D_t\ \tilde{f}(x,t) = \Big[\ \frac{D_x^2}{2} + \sum_{n\geq 2} c_n\ \frac{D_x^n}{n}\Big]\ \tilde{f}(x,t)$$

with the formal solution

$$\tilde{f}(x,t) = e^{t\frac{D_x^2}{2}}\ \exp\Big[\ t\sum_{n\geq 2} c_n\ \frac{D_x^n}{n}\Big]\ f(x).$$

With $x = y\sqrt{t}$,

$$e^{t\frac{D_x^2}{2}} \exp[\, t \sum_{n \geq 2} c_n \frac{D_x^n}{n} \,] f(x) = e^{\frac{D_y^2}{2}} \exp[\, \sum_{n \geq 2} t^{1-\frac{n}{2}} c_n \frac{D_y^n}{n} \,] f(y\sqrt{t})$$

$$= e^{h.D_y} \exp[\, \sum_{n \geq 2} \gamma_n D_y^n \,] f(y\sqrt{t})$$

$$= e^{h.D_y} e^{St1.(0,\gamma_2,\gamma_3,...;1)\, D_y} f(y\sqrt{t}) = e^{[h.+St1.(0,\gamma_2,\gamma_3,...;1)]\, D_y} f(y\sqrt{t})$$

$$= e^{St1.(h.,\gamma_2,\gamma_3,...;1)\, D_y} f(y\sqrt{t}) = e^{H.[St1.(0,\gamma_2,\gamma_3,...;1)]\, D_y} f(y\sqrt{t})$$

$$= f[[y + h. + St1.(0, \gamma_2, \gamma_3, ...; 1)]\sqrt{t}]$$

$$= f[[h. + St1.(y, \gamma_2, \gamma_3, ...; 1)]\sqrt{t}] = f[St1.(y + h., \gamma_2, \gamma_3, ...; 1)\sqrt{t}]$$

$$= f[H.[St1.(y, \gamma_2, \gamma_3, ...; 1)]\sqrt{t}] = f[St1.(H.(y), \gamma_2, \gamma_3, ...; 1)\sqrt{t}]$$

with $\gamma_n = t^{1-\frac{n}{2}} c_n$ for $n > 1$ and, again, $y = x/\sqrt{t}$.

With $f(x) = x^n$, this gives

$$\tilde{f}(x, t) = H_n[St1.(y, \gamma_2, \gamma_3, ...; 1)]/t^{\frac{n}{2}} = St1_n(H.(y), \gamma_2, \gamma_3, ...; 1)/t^{\frac{n}{2}}.$$

In particular, for $t = 1$,

$$\tilde{f}(x, 1) = H_n[St1.(x, c_2, c_3, ...; 1)] = St1_n(H.(x), c_2, c_3, ...; 1).$$

For $c_n = 0$ for $n > 1$, this reduces to the Hermite polynomials.

Alternatively, with $\bar{c}_2 = 1 + c_2$,

$$\tilde{f}(x,t) = e^{t\frac{D_x^2}{2}} \exp[\, t \sum_{n\geq 2} c_n \, \frac{D_x^n}{n} \,]\, f(x) = \exp[\, t \sum_{n\geq 2} \tilde{c}_n \, \frac{D_x^n}{n} \,]\, f(x)$$

$$= e^{St1.(0,t\,\bar{c}_2,t\,c_3,...;1)\, D_x}\, f(x) = f[x + St1.(0,t\,\bar{c}_2, t\, c_3, ...; 1)]$$

$$= f[St1.(x, t\bar{c}_2, tc_3, ...; 1)] = f[St1.(\frac{x}{t}, \bar{c}_2, c_3, ...; t)].$$

Reprising, the deformed heat/diffusion equation

$$D_t\, \tilde{f}(x,t) = [\, \frac{D_x^2}{2} + \sum_{n\geq 2} c_n \, \frac{D_x^n}{n} \,]\, \tilde{f}(x,t)$$

has the solution

$$\tilde{f}(x,t) = e^{t\frac{D_x^2}{2}} \exp[\, t \sum_{n\geq 2} c_n \, \frac{D_x^n}{n} \,]\, f(x)$$

$$= f[H.[St1.(y, \gamma_2, \gamma_3, ...; 1)]\sqrt{t}] = f[St1.(H.(y), \gamma_2, \gamma_3, ...; 1)\sqrt{t}]$$

$$= f[St1.(x, t\bar{c}_2, tc_3, ...; 1)] = f[(x + t \sum_{n\geq 1} \bar{c}_{n+1}\, D_x^n)]\, 1$$

with
$\gamma_n = t^{1-\frac{n}{2}}\, c_n$,

$y = x/\sqrt{t}$,

$\bar{c}_2 = 1 + c_2$,

$St1_n(x, c_2, ...; 1) = (x + \sum_{n\geq 1} c_{n+1}\, D_x^n)\, 1$,

and initial condition

$\tilde{f}(x,0) = f(x).$

Since these umbral compositions can be represented by multiplication of invertible lower triangular matrices, conjugations (or 'axes rotations') can be used to map between the nondeformed and deformed reps.

Checks:

For $n = 1$,

$$S_1[\tfrac{x}{t} + h.; t] = S_1[H.(\tfrac{x}{t}); 1] = H_1[St1.(\tfrac{x}{t}), c_2, c_3, ...; t]$$

$$= (\tfrac{x}{t} + h.)t = x + tH_1(\tfrac{x}{t}).$$

For $n = 3$,

$$St1_3(d_1, d_2, d_3; 1) = 2d_3 + 3d_1 d_2 + d_1^3,$$

$h_0 = h_2 = 1$, and $h_1 = h_3 = 0$, so

$$H_1(x) = (h. + x)^1 = \sum_{k=0}^{1} \binom{1}{k} h_k\, x^{1-k} = x,$$

$$H_2(x) = (h. + x)^2 = \sum_{k=0}^{2} \binom{2}{k} h_k\, x^{2-k} = x^2 + 1,$$

$$H_3(x) = x^3 h_0 + 3x^2 h_1 + 3x h_2 + h_3 = x^3 + 3x,$$

and

$$H_3[St1.(y, \gamma_2, \gamma_3, ...; 1)]t^{\frac{3}{2}} = [St1_3(y, \gamma_2, \gamma_3; 1) + 3S_1(y; 1)]t^{\frac{3}{2}}$$

$$= [(2\gamma_3 + 3\gamma_2 y + y^3) + 3y]t^{\frac{3}{2}} = 2c_3 t + 3c_2 xt + x^3 + 3xt$$

while

$$St1_3(H.(y), \gamma_2, \gamma_3; 1)t^{\frac{3}{2}} = [2\gamma_3 + 3H_1(y)\gamma_2 + H_3(y)]t^{\frac{3}{2}} = 2c_3 t + 3xc_2 t + x^3 + 3xt$$

$$= St1_3(x, t\bar{c}_2, tc_3; 1) = 2c_3 t + 3x(1 + c_2)t + x^3 = [2c_3 + 3x(1 + c_2)]t + x^3.$$

Then for

$$f(x) = x^3,$$

$$\tilde{f}(x, t) = [2c_3 + 3x(1 + c_2)]t + x^3,$$

$$D_t \; \tilde{f}(x,t) = 2c_3 + 3(1 + c_2)x$$

and

$$[\; \tfrac{D_x^2}{2} + \textstyle\sum_{n \geq 2} \; c_n \; \tfrac{D_x^n}{n} \;] \; \tilde{f}(x,t)$$

$$= [\; (1 + c_2) \; \tfrac{D_x^2}{2} + c_3 \; \tfrac{D_x^3}{3} \;] \; [x^3 + (2c_3 + 3(1 + c_2)x)t]$$

$$= 3(1 + c_2)x + 2c_3,$$

giving consistent results.

Normal ordering of the operator $\mathfrak{R}^n = (R + L)^n$

With L and R the ladder ops of $p_n(x)$,

$$(L + R) \; p_0(x) = (L + R) \; 1 = 0 + p_1(x) = p_1(x) \; ,$$

$$(L + R) \; p1(x) = p_0(x) + p_1(x) = 1 + p_2(x) \; ,$$

$$(L + R) \; 1 + p_2(x) = 0 + 2p_1(x) + p_1(x) + p_3(x) = 3p_1(x) + p_3(x) \; ,$$

$$(L + R) \; 3p_1(x) + p_3(x) = 3 + 3p_2(x) + 3p_2(x) + p_4(x) = 3 + 6p_2(x) + p_4(x) \; ,$$

$$(L + R) \; 3 + 6p_2(x) + p_4(x) = 12p_1(x) + 4p_3(x) + 3p_1(x) + 6p_3(x) + p_5(x)$$

$$= 15p_1(x) + 10p_2(x) + p_5(x) \; ,$$

consistent with

$$(R + L)^n 1 = H_n(p.(x)) = (h. + p.(x))^n = P_n(x).$$

$\mathbb{R} = R + L$ is the raising op for the polynomials $P_n(x) = H_n(p, (x))$. The e.g.f. for $p_n(x)$ a Sheffer sequence $S_n(x)$ with the ladder ops R and L and e.g.f. $e^{S.(x)t} = A(t)e^{xB(t)}$ is determined by

$$e^{P.(x)t} = e^{t(R+L))} \; 1 = e^{\frac{t^2}{2}} \; e^{tR} \; e^{tL} \; 1$$

$$= e^{\frac{t^2}{2}} \; e^{tR} \; 1 = e^{\frac{t^2}{2}} \; e^{S.(x)t} = e^{\frac{t^2}{2}} \; A(t) \; e^{xB(t)},$$

$$= e^{h.t} e^{S.(x)t} = e^{(h.+S.(x))t} = e^{H.(S.(x))t}.$$

Then

$$\mathbb{S}_n(x) = P_n(x) = H_n(S.(x))$$

is also a Sheffer sequence with $\mathbb{A}(t) = e^{\frac{t^2}{2}} \; A(t)$ and the ladder ops and e.g.f

$$\mathbb{R} = R + L \; , \;\; \mathbb{L} = L \; , \;\; e^{S.t} = e^{\frac{t^2}{2}} \; A(t) \; e^{xB(t)} = \mathbb{A}(t) \; e^{xB(t))}.$$

To express $(x + D_x)^n$ in terms of summands of the form $x^k \; D_x^m$ translate the noncommutative operations into commutative operations using the ODR as

$$e^{-xy} \; e^{t(x+D_x)} \; e^{xy} = e^{-xy} \; e^{\frac{t^2}{2}} \; e^{tx} e^{tD_x} e^{xy}$$

$$= e^{\frac{t^2}{2}} \; e^{t(x+y)} = e^{(h.+x+y)t} = e^{(H.(x)+y)t} = e^{(x+H.(y))t} = e^{H.(x+y)t},$$

and, reflecting

$$H_n(x + y) = (H.(x) + y)^n = \sum_{k=0}^{n} \binom{n}{k} H_k(x) \; y^{n-k},$$

$$(x + D_x)^n = \sum_{k=0}^{n} \binom{n}{k} H_k(x) \; D_x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (h. + x)^k \; D_x^{n-k}$$

and

$$\mathbb{R}^n = (R + L)^n = \sum_{k=0}^{n} \binom{n}{k} H_k(R) \; L^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (h. + R)^k \; L^{n-k}.$$

Then for $\mathbb{R}^n = (R + L)^n$, the coefficient of $R^k \; L^m$ is

$$C_{n,k,m} = \frac{D_{x=0}^k}{k!} \; \frac{D_{y=0}^m}{m!} \; (h.+x+y)^n$$

$$= \frac{n!}{m!} \; \frac{D_{x=0}^k}{k!} \; \frac{(h.+x)^{n-m}}{(n-m)!} = \frac{n!}{k!\,m!} \; \frac{h_{n-k-m}}{(n-k-m)!}.$$

From above, the bivariate e.g.f. for the rows of coefficients for the normal ordering of $(R+L)^n$ is

$$e^{\frac{t^2}{2}} \; e^{t(x+y)} = e^{h.t} \; e^{t(x+y)} = e^{(h.+x+y)t} = e^{(H.(x)+y)t} = e^{(x+H.(y))t} = e^{H.(x+y)t}$$

with bivariate row o.g.f.s given by

$$(h.+x+y)n = (H.(x)+y)^n = (x+H.(y))^n = H_n(x+y) = \sum_{k=0}^n \binom{n}{k} h_{n-k} \, (x+y)^k,$$

a weighting of alternating rows of the Pascal matrix [A007318](#) from one perspective since h_m vanishes for odd m

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Geometric combinatorics--pair matchings in hypertetrahedrons

The moments for the Hermite polynomials are the sequence of h_n, the aerated odd double factorials OEIS

[A001147](#) $= 1, \; 0, \; 1, \; 0, \; 3, \; 0, \; 15, \; 0, \; 105, \ldots, \; 0, \; (1)(3)(5)\cdots(2m-1),\ldots$,

enumerating the perfect pair matchings of the $(n-1)$-dimensional simplex with n vertices, the $(n-1)$-D hypertriangle, or hypertetrahedron. An $(n-1)$-D hypertetrahedron has a perfect matching when each of its n vertices is included in a colored edge (gray = uncolored) and no colored edge touches another colored edge, i.e., the colored edges are disjoint, or independent. In other words, a perfect matching is a complete disjoint pairing of the vertices of the hypertetrahedron. A useful convention is to assign the identity to the nominal number of perfect matchings of the (-1)-D simplex, with no vertices. For example, the triangle has no perfect matchings, so $h_3 = 0$, likewise for any even dimensional hypertetrahedron, which has an odd number of vertices. The tetrahedron has three distinct perfect matchings (draw them).

The Hermite polynomial

$$H_n(x) = (h. + x)^n \sum_{k=0}^{n} h_k \, x^{n-k}$$

is an ordinary generating function for the number of different ways that an $(n-1)$-D hypertetrahedron, with n vertices, can be formed from colored $(k-1)$-D tetrahedrons, with k vertices, with perfect matchings and uncolored (gray, or unmatched) $(n-k-1)$-D tetrahedrons, with $(n-k)$ vertices. This interpretation is reflected in the coefficient of x^n being $h_0 = 1$ since it corresponds simply to one gray tetrahedron while the coefficient of x^0 is h_n, the number of perfect matchings of a colored $(n-1)$-D hypertetrahedron, with n vertices. An equivalent perspective is to consider an $(n-1)$-D hypertetrahedron with k gray vertices while the other $(n-k)$ (when an even integer) are colored, being the endpoints of disjoint colored edges, so it has $(n-k)/2$ independent pairings while k vertices, all labeled with x remain unpaired.

The interpretation of the coefficients of $H_n(x+y) = (h. + x + y)^n$ is a further refinement of this simplex model with the uncolored vertices being labeled with either an x or y. For example,

$$(x+D)^2 = xx + xD + Dx + D^2 = x^2 + 2xD + D^2 + 1$$

corresponds to

$$H_n(x+y) = (h. + x + y)^2 = h_0(x+y)^2 + 2h_1 \cdot (x+y) + h_2 = h_0 \, (x+y)^2 + h_2 \, x^0 \, y^0$$

$$= x^2 + 2xy + y^2 + 1,$$

which, in turn, corresponds to a line segment, the 1-D simplex, with each vertex labeled with an x; or one with an x, the other a y; or with each labeled with a y; or one unlabeled, colored matched pair.

The same coefficients must result from normal ordering using the Graves-Pincherle-Heisenberg-Weyl commutator $[L, R] = 1$, or $LR = RL + 1$ of, e.g., the permutations represented by

$$(L+R)^2 = LL + LR + RL + RR$$

and

$$(L+R)^3 = LLL + LLR + LRL + LRR + RLL + RLR + RRL + RRR.$$

See A344678 for other models, connections to other OEIS arrays, and references to earlier work.

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Related Stuff:

See my previous post on the Hermite polynomials and the OEIS entries for more info.

See my next post on the analytic and combinatorial relations among the Catalan and the odd double factorials, containing refs on relations among orthogonal polynomials (the Hermite being a fundamental classic example), compositional and multiplicative inversion, Lie group theoretics, continued fractions, cumulants, lattices, Riccati equations, and random walks.